



**SECONDARY MATHEMATICS BENCHMARKS
PROGRESSIONS
GRADES 7 – 12**

Achieve, Inc.

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Secondary Mathematics Benchmarks Progressions Grades 7–12

Topics are arranged in five content strands (Number, Discrete Mathematics, Algebra, Geometry, and Probability and Statistics) and five levels (A, B, C, D, and E) representing increasing degrees of mathematical complexity within each content strand. Together, the five levels include content that spans *roughly* grades 7 through 12, although they do not directly correspond to grade levels. Levels A–D correspond to the mathematical expectations appropriate for all high school graduates; Level E comprises a subset of elective topics that are important for some but not all postsecondary endeavors. Expectations are grouped into small, coherent content clusters; each content cluster is placed at the earliest level at which it might be addressed.

Number (N)

Number sense is the cornerstone for mathematics in everyday life. Comparing prices, deciding whether to buy or lease a car, estimating tax on a purchase or tip for a service, and evaluating salary increases in the context of annual inflation rates all require understanding of and facility with quantified information. Interpreting much of what appears in daily news releases relies on an ability to glean valid information from numerical data and evaluate claims based on data. Through the study and application of ratios, rates, and derived measures, students extend their sense of number to contextual situations, paying heed to units. They develop the capacity to work with precision and accuracy and to spot and minimize errors, an important skill in a world that increasingly relies on quality control. In all of these endeavors, electronic technology provides an accurate and efficient way to manage quantitative information. In addition, fluency and flexibility in the algorithms and properties that govern numerical operations are important for procedural computation; they lay conceptual foundations for the study of algebra and for reasoning in all areas of mathematics.

N.A.1 Rational numbers

a. Identify rational numbers, represent them in various ways, and translate among these representations.

Rational numbers are those that can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

- Identify whole numbers, fractions (positive and negative), mixed numbers, finite (terminating) decimals, and repeating decimals as rational numbers.

The decimal form of a rational number eventually repeats. A decimal is called terminating if its repeating digit is 0. A fraction has a terminating decimal expansion only if its denominator in reduced form has only 2 and 5 as prime factors.

- Express a percent having a finite number of digits as a rational number by expressing it as a ratio whose numerator is an integer and whose denominator is 100 (or, more generally, whose denominator is a power of 10).
- Transform rational numbers from one form (fractions, decimals, percents and mixed numbers) to another.

b. Understand and use inequalities to compare rational numbers; apply basic rules of inequalities to transform expressions involving rational numbers.

For any rational numbers a , b , and c , $a < b$ implies that $a + c < b + c$; further, $a < b$ implies that $-a > -b$. For $c > 0$, $a < b$ implies that $ac < bc$; for $c < 0$, $a < b$ implies that $ac > bc$.

c. Locate rational numbers on the number line and explain the significance of these locations.

- Show that a number and its opposite are mirror images with respect to the point 0.
- Identify one or more rational numbers that lie between two given rational numbers and explain how this can be done no matter how close together the given numbers are.

d. Know and apply effective methods of calculation with rational numbers.

- Demonstrate understanding of algorithms for addition, subtraction, multiplication, and division of numbers expressed as fractions, terminating decimals, or repeating decimals by applying the algorithms and explaining why they work.
- Add, subtract, multiply, and divide (with a non-zero divisor) rational numbers and explain why these operations always produce another rational number.
- For any rational number r , determine its opposite, $-r$ and its reciprocal, $1/r$, if $r \neq 0$. Explain why $-r$ and $1/r$ are rational whenever r is rational.
- Extend the properties of computation with whole numbers (e.g., commutative property, associative property, distributive property) to rational number computation.
- Interpret parentheses and employ conventional order of operations in a numerical expression.
- Check answers by estimation or by independent calculations, with or without calculators and computers.

e. Recognize, describe, extend, and create well-defined numerical patterns.

Of special interest are arithmetic sequences, those generated by repeated addition of a fixed number, and geometric sequences, those generated by repeated multiplication of a fixed number.

f. Solve practical problems involving rational numbers.

Examples: Calculate markups, discounts, taxes, tips, average speed.

N.A.2 Absolute values

a. Know and apply the definition of absolute value.

The absolute value is defined by $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

b. Interpret absolute value as distance from zero.

c. Interpret absolute value of a difference as "distance between."

Example: $|5 - 1| = 4$ is the distance between 5 and 1 on the number line.

N.A.3 Prime decomposition, factors, and multiples

a. Know and apply the Fundamental Theorem of Arithmetic, that every positive integer is either prime itself or can be written as a unique product of primes (ignoring order).

- Identify prime numbers; describe the difference between prime and composite numbers.
- Determine divisibility rules, use them to help factor composite numbers, and explain why they work.
- Write a prime decomposition for numbers up to 100.

b. Explain the meaning of the greatest common divisor (greatest common factor) and the least common multiple and use them in operations with fractions.

- Determine the greatest common divisor and least common multiple of two whole numbers from their prime factorizations.
- Use greatest common divisors to reduce fractions $\frac{n}{m}$ and ratios $n:m$ to an equivalent form in which $\text{gcd}(n, m) = 1$.

Fractions $\frac{n}{m}$ in which $\text{gcd}(n, m) = 1$ are said to be in lowest terms.

- Add and subtract fractions by using least common multiple of denominators.

c. Write equivalent fractions by multiplying both numerator and denominator by the same non-zero whole number or dividing by common factors in the numerator and denominator.

N.A.4 Ratio, rates, and derived quantities

a. Interpret and apply measures of change such as percent change and rates of growth.

b. Calculate with quantities that are derived as ratios and products.

- Interpret and apply ratio quantities including velocity, density, pressure, population density.
Examples of units: Feet per second, grams per cc^3 , people per square mile.
- Interpret and apply product quantities including area, volume, energy, work.
Examples of units: Square meters, kilowatt hours, person days.

c. Solve data problems using ratios, rates, and product quantities.

- Convert measurements both within and across measurement systems.

d. Create and interpret scale drawings as a tool for solving problems.

e. Use unit analysis to clarify appropriate units in calculations.

Example: The calculation for converting 50 feet per second to miles per hour can be checked using the unit calculation. $\frac{50 \text{ feet}}{1 \text{ second}} \times \frac{1 \text{ mile}}{5280 \text{ feet}} \times \frac{60 \text{ seconds}}{1 \text{ minute}} \times \frac{60 \text{ minute}}{1 \text{ hour}} = 34.09 \frac{\text{miles}}{\text{hour}}$ yields the correct units since the units feet, seconds, and minutes all appear in both numerator and denominator.

f. Identify and apply derived measures.

Derived measures are quantities determined by calculation.

Examples: Percent change, density, the composite scale used for college rankings.

g. Use and identify potential misuses of weighted averages.

- Identify and interpret common instances of weighted averages.
Examples: Grade averages, stock market indexes, Consumer Price Index, unemployment rate.
- Analyze variation in weighted averages and distinguish change due to weighting from changes in the quantities measured.

Example: Suppose a company employed 100 women with average annual salaries of \$20,000 and 500 men with average salaries of \$40,000. After a change in management, the company employed 200 women and 400 men. To correct past inequities, the new management increased women's salaries by 25% and men's salaries by 5%. Despite these increases, the company's average salary declined by almost 1%.

N.B.1 Estimation and approximation

a. Use simple estimates to predict results and verify the reasonableness of calculated answers.

- Use rounding, regrouping, percentages, proportionality, and ratios as tools for mental estimation.

b. Develop, apply, and explain different estimation strategies for a variety of common arithmetic problems.

Examples: Estimating tips, adding columns of figures, estimating interest payments, estimating magnitude.

c. Explain the phenomenon of rounding error, identify examples, and, where possible, compensate for inaccuracies it introduces.

- Interpret apportionment as a problem of fairly distributing rounding error.
Examples: Analyzing apportionment in the U.S. House of Representatives; creating data tables that sum properly.

d. Determine a reasonable degree of precision in a given situation.

- Assess the amount of error resulting from estimation and determine whether the error is within acceptable tolerance limits.
- Choose appropriate techniques and tools to measure quantities in order to achieve specified degrees of precision, accuracy, and error (or tolerance) of measurements.

Example: Humans have a reaction time to visual stimuli of approximately 0.1 sec. Thus, it is reasonable to use hand-activated stopwatches that measure tenths of a second but not hundredths.

e. Interpret and compare extreme numbers (e.g., lottery odds, national debt, astronomical distances).

f. Apply significant figures, orders of magnitude, and scientific notation when making calculations or estimations.

g. In a problem situation, use judgment to determine when an estimate is appropriate and when an exact answer is needed.

N.B.2 Exponents and roots

a. Use the definition of a root of a number to explain the relationship of powers and roots.

If $a^n = b$, for an integer $n \geq 0$, then a is said to be an n th root of b . When n is even and $b > 0$, we identify the unique $a > 0$ as the principal n th root of b , written $\sqrt[n]{b}$.

- Use and interpret the symbols $\sqrt{\quad}$ and $\sqrt[3]{\quad}$; know that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, $(\sqrt{a})^2 = a$, $\sqrt{a^2} = |a|$, and $\sqrt[3]{a^3} = a$.

By convention, for $a > 0$, \sqrt{a} is used to represent the non-negative square root of a .

b. Estimate square and cube roots and use calculators to find good approximations.

- Know the squares of numbers from 1 to 12 and the cubes of numbers from 1 to 5.
- Make or refine an estimate for a square root using the fact that if $0 \leq a < n < b$, then $0 \leq \sqrt{a} < \sqrt{n} < \sqrt{b}$; make or refine an estimate for a cube root using the fact that if $a < n < b$, then $\sqrt[3]{a} < \sqrt[3]{n} < \sqrt[3]{b}$.

c. Evaluate expressions involving positive integer exponents and interpret such exponents in terms of repeated multiplication.

d. Convert between forms of numerical expressions involving roots and perform operations on numbers expressed in radical form.

Example: Convert $\sqrt{8}$ to $2\sqrt{2}$ and use the understanding of this conversion to perform similar calculations and to compute with numbers in radical form.

e. Interpret rational and negative exponents and use them to rewrite expressions in alternative forms.

Examples: $3^{-2} = \frac{1}{9}$; $5^{\frac{3}{2}} = \sqrt{5^3} = 5\sqrt{5}$.

- Convert between expressions involving rational exponents and those involving roots and integral powers.

Examples: $5^{\frac{3}{2}} = \sqrt{5^3} = 5\sqrt{5}$; $\sqrt[4]{27} = \sqrt[4]{3^3} = 3^{\frac{3}{4}}$.

- Convert between expressions involving negative exponents and those involving only positive ones.

Examples: $3^{-2} = \frac{1}{9}$; $\frac{2^{-3}}{7^{-1}} = \frac{7}{2^3} = \frac{7}{8}$.

- Apply the laws of exponents to expressions containing rational exponents.

Examples: $3^{\frac{2}{3}} = \left(3^2\right)^{\frac{1}{3}} = 9^{\frac{1}{3}}$; $\left(2^3\right)^{\frac{1}{4}} = 2^{\frac{3}{4}}$; $\frac{5^{\frac{4}{3}}}{5^{\frac{1}{3}}} = 5$.

N.B.3 Real numbers

a. Categorize real numbers as either rational or irrational and know that, by definition, these are the only two possibilities.

- Locate any real number on the number line.
- Apply the definition of irrational numbers to identify examples and recognize approximations.

Square roots, cube roots, and n^{th} roots of whole numbers that are not respectively squares, cubes, and n^{th} powers of whole numbers provide the most common examples of irrational numbers. Pi (π) is another commonly cited irrational number.

- Know that the decimal expansion of an irrational number never ends and never repeats.
- Recognize and use $\frac{22}{7}$ and 3.14 as approximations for the irrational number represented by pi (π).
- Determine whether the square, cube, and n^{th} roots of integers are integral or irrational when such roots are real numbers.

b. Establish simple facts about rational and irrational numbers using logical arguments and examples.

Examples: Explain why, if r and s are rational, then both $r + s$ and rs are rational, for example, both $\frac{3}{4}$ and 2.3 are rational; $\frac{3}{4} + 2.3 = \frac{3}{4} + \frac{23}{10} = \frac{15}{20} + \frac{46}{20} = \frac{61}{20}$, which is the ratio of two integers, hence rational; give examples to show that, if r and s are irrational,

then $r + s$ and rs could be either rational or irrational, for example, $\sqrt{3} + \frac{\sqrt{3}}{2}$ is irrational whereas $(5 + \sqrt{2}) - \sqrt{2}$ is rational.

c. Show that a given interval on the real number line, no matter how small, contains both rational and irrational numbers.

Example: To determine an irrational number between $\sqrt{10}$ and $3\frac{1}{3}$, consider that $\sqrt{10} \approx 3.1622776 \dots$, while $3\frac{1}{3} = 3.\bar{3}$ so the number $3.17177177717 \dots$; where the number of 7s in each successive set of 7s increases by one, is irrational and lies in this interval.

- Given a degree of precision, determine a rational approximation to that degree of precision for an irrational number.

d. Extend the properties of computation with rational numbers to real number computation.

Example: If the area of one circle is $4p$ and the area of another, disjoint circle is $25p$, then the sum of the areas of the two circles is $4p + 25p = (4 + 25)p = 29p$, since the distributive property is true for all real numbers.

N.C.1 Number bases

a. Identify key characteristics of the base-10 number system and adapt them to other common number bases (binary, octal, and hexadecimal).

- Represent and interpret numbers in the binary, octal, and hexadecimal number systems.
- Apply the concept of base-10 place value to understand representation of numbers in other bases.

Example: In the base-8 number system, the 5 in the number 57,273 represents 5×8^4 .

b. Convert binary to decimal and vice versa.

c. Encode data and record measurements of information capacity using various number base systems.

N.D.1 Complex numbers

a. Know that if a and b are real numbers, expressions of the form $a + bi$ are called complex numbers, and explain why every real number is a complex number.

Every real number, a , is a complex number because it can be expressed as $a + 0i$.

The imaginary unit, sometimes represented as $i = \sqrt{-1}$, is a solution to the equation $x^2 = -1$.

- Express the square root of a negative number in the form bi , where b is real.

Just as with square roots of positive numbers, there are two square roots for negative numbers; in $\sqrt{-4} = \pm 2i$, $2i$ is taken to be the principal square root based on both the Cartesian and trigonometric representations of complex numbers.

Examples: Determine the principle square root for each of the following:

$$\sqrt{-7} = i\sqrt{7}; \quad \sqrt{-256} = 16i.$$

b. Identify complex conjugates.

The conjugate of a complex number $a + bi$ is the number $a - bi$.

c. Determine complex number solutions of the form $a + bi$ for certain quadratic equations.

- Know that complex solutions of quadratic equations with real coefficients occur in conjugate pairs and show that multiplying factors related to conjugate pairs results in a quadratic equation having real coefficients.

Example: The complex numbers $(3 + i\sqrt{5})$ and $(3 - i\sqrt{5})$ are the roots of the equation $(x - (3 + i\sqrt{5}))(x - (3 - i\sqrt{5})) = x^2 - 6x + 14 = 0$, whose coefficients are real.

N.E.1 Computation with complex numbers

a. Compute with complex numbers.

- Add, subtract, and multiply complex numbers using the rules of arithmetic.
- Use conjugates to divide complex numbers.

Example: $\frac{(5 + 4i)}{3 - 2i} = \frac{(5 + 4i)}{(3 - 2i)} \cdot \frac{(3 + 2i)}{(3 + 2i)} = \frac{15 + 22i + 8i^2}{9 - 4i^2} = \frac{7 + 22i}{13}$ or $\frac{7}{13} + \frac{22}{13}i$.

This process can also be applied to the division of irrational numbers involving square roots, such as $a + \sqrt{b}$ and $a - \sqrt{b}$.

N.E.2 Argand diagrams

a. Interpret complex numbers graphically using an Argand diagram.

In an Argand diagram, the real part of a complex number $z = x + iy$ is plotted along the horizontal axis, and the imaginary part is plotted on the vertical axis. An Argand diagram enables complex numbers to be plotted as points in the plane just as the real line enables real numbers to be plotted as points on a line.

b. Represent the complex number $z = x + iy$ in the polar form $z = r(\cos\theta + i\sin\theta)$ and interpret this form graphically, identifying r and θ .

c. Explain the effect of multiplication and division of complex numbers using an Argand diagram and its relationship to the polar form of a complex number.

Discrete Mathematics (D)

Discrete mathematics, sometimes called finite mathematics, can be thought of as the science of counting, arrangements, and algorithms. It offers a plethora of concrete, practical problems (e.g., fair apportionment, searching algorithms, error-correction methods) and a wealth of subtle problems whose statements are deceptively simple but whose solutions provide significant challenge. While events in the physical world are most often modeled by continuous mathematics (i.e., the calculus and prerequisite topics in algebra, geometry, and trigonometry), the increasingly important world of computers, information technology, and logistics employs a different type of mathematics. New approaches and applications require the use of discrete processes, many of which have not traditionally been included in core high school courses. To be well prepared for the future, all students need to understand the concepts and applications of this important area of mathematics.

D.A.1 Sets and Boolean Algebra

a. Know the concepts of sets, elements, empty set, relations (e.g., belong to), and subsets, and use them to represent relationships among objects and sets of objects.

- Recognize and use different methods to define sets (lists, defining property).

b. Perform operations on sets: union, intersection, complement.

Example: Use Boolean search techniques to refine online bibliographic searches.

c. Create and interpret Venn diagrams to solve problems.

d. Identify whether a given set is finite or infinite; give examples of both finite and infinite sets.

D.B.1 Permutations and combinations

a. Determine the number of ways events can occur using permutations, combinations, and other systematic counting methods.

A permutation is a rearrangement of distinct items in which their order matters; a combination is a selection of a given number of distinct items from a larger number without regard to their arrangement (i.e., in which their order does not matter).

- Know and apply organized counting techniques such as the Fundamental Counting Principle.

The Fundamental Counting Principle is a way of determining the number of ways a sequence of events can take place. If there are n ways of choosing one thing and m ways of choosing a second after the first has been chosen, then the Fundamental Counting Principle says that the total number of choice patterns is $n \cdot m$.

Examples: How many different license plates can be formed with two letters and three numerals? If the letters had to come first, how many letters would be needed to create at least as many different license plate numbers? How many different subsets are possible for a set having six elements?

- Distinguish between situations that do not permit replacement and situations that do permit replacement.

Examples: How many different four-digit numbers can be formed if the first digit must be non-zero and each digit may be used only once? How many are possible if the first digit must be non-zero but digits can be used any number of times?

- Distinguish between situations where order matters and situations where it does not; select and apply appropriate means of computing the number of possible arrangements of the items in each case.

b. Interpret and simplify expressions involving factorial notation.

Examples: Interpret $6!$ as the product $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$; recognize that

$$\frac{15!}{12!} = 15 \cdot 14 \cdot 13 = 2,730.$$

D.B.2 Discrete Graphs

a. Construct and interpret decision trees.

A tree is a connected graph containing no closed loops (cycles).

- Represent and analyze possible outcomes of independent events (e.g., repeated tossing of a coin, or throwing dice) using tree diagrams.

Tree diagrams can also be used to analyze games such as tic-tac-toe or Nim or simply to organize outcomes.

b. Create and interpret network graphs.

A graph is a collection of points (nodes) and the lines (edges) that connect some subset of those points; a cycle on a graph is a closed loop created by a subset of edges. A directed graph is one with one-way arrows as edges.

- Use graphs to diagram and study social and organizational networks.

Examples: Determine the shortest route for recycling trucks; schedule when contestants play each other in a tournament; illustrate all possible travel routes that include four cities; interpret a directed graph to determine the result of a tournament.

c. Construct and interpret flow charts.

D.B.3 Iteration and recursion

a. Analyze and interpret relationships represented iteratively and recursively.

Example: Recognize that the sequence defined by "First term = 5. Each term after the first is six more than the preceding term" is the sequence whose first seven terms are 5, 11, 17, 23, 29, 35, and 41.

- Analyze the sequences produced by recursive calculations using spreadsheets.

Example: The result of repeatedly squaring a number between -1 and 1 appears to approach zero, while the result of repeatedly squaring a number less than -1 or

greater than 1 appears to continue to increase; determine empirically how many steps are needed to produce 4-digit accuracy in square roots by iterating the operations *divide* and *average*.

- Describe the factorial function or the *Fibonacci sequence* recursively.

b. Generate and describe sequences having specific characteristics.

- Generate a relatively small number of terms by hand and use calculators and spreadsheets effectively to extend the sequence.
- Describe arithmetic sequences recursively.

Arithmetic sequences are those in which each term differs from its preceding term by a constant difference. To describe an arithmetic sequence, both the starting term and the constant difference must be specified.

- Describe geometric sequences recursively.

Geometric sequences are those in which each term is a constant multiple of the term that precedes it. To describe a geometric sequence, both the starting term and the constant multiplier (often called the common ratio) must be specified.

- Given an irrational number expressed using rational exponents or radicals, find increasing and decreasing sequences that converge to that number and show that the first terms of these sequences satisfy the right inequalities.

Example: $1 < 1.4 < 1.41 < 1.414 < \dots < \sqrt{2} < 1.415 < 1.42 < 1.5 < 2$ since

$$1^2=1 < (1.4)^2=1.96 < (1.41)^2=1.9881 < (1.414)^2=1.999396 < \dots < (\sqrt{2})^2=2 < \dots < (1.415)^2=2.002225 < (1.42)^2=2.0164 < (1.5)^2=2.25 < 2^2=4$$

- Describe other regular patterns of growth recursively.
 Examples: The factorial function; the Fibonacci sequence.

c. Demonstrate the effect of compound interest, decay, or growth using iteration.

- Display the effect of iteration using a calculator or spreadsheet.
 Examples: Enter the amount of a loan, the monthly interest rate, and the monthly payment in a spreadsheet. The formula $(\text{loan amount}) \cdot (1 + \text{interest rate}) - (\text{monthly payment})$ gives the amount remaining monthly on the loan at the end of the first month, and the iterative “fill down” command will show the amount remaining on the loan at the end of each successive month; a similar process using past data about the yearly percent increase of college tuition and annual inflation rate will provide an estimate of the cost of college for a newborn in current dollar equivalents.
- Identify the diminishing effect of increasing the number of times per year that interest is compounded and relate this to the notion of instantaneous compounding

D.C.1 Algorithms

a. Identify and give examples of simple algorithms.

An algorithm is a procedure (a finite set of well-defined instructions) for accomplishing some task that, given an initial state, will terminate in a defined end-state. Recipes and assembly instructions are everyday examples of algorithms.

- Analyze and compare simple computational algorithms.
Examples: Write the prime factorization for a large composite number; determine the least common multiple for two positive integers; identify and compare mental strategies for computing the total cost of several objects.
- Analyze and apply the iterative steps in standard base-10 algorithms for addition and multiplication of numbers.

b. Analyze and apply algorithms for searching, for sorting, and for solving optimization problems.

- Identify and apply algorithms for searching, such as sequential and binary.
- Describe and compare simple algorithms for sorting, such as bubble sort, quick sort, and bin sort.
Example: Compare strategies for alphabetizing a long list of words; describe a process for systematically solving the Tower of Hanoi problem.
- Know and apply simple optimization algorithms.
Example: Use a vertex-edge graph (network diagram) to determine the shortest path for accomplishing some task.

D.C.2 Mathematical reasoning

a. Use correct mathematical notation, terminology, syntax, and logic.

- Explain reasoning in both oral and written forms.

b. Distinguish between inductive and deductive reasoning.

Inductive reasoning should be clearly distinguished from the deductive mathematical reasoning involved in mathematical induction.

- Identify inductive reasoning as central to the scientific method and deductive reasoning as characteristic of mathematics.

Inductive reasoning is based on observed patterns and can be used in mathematics to generate conjectures, after which deductive reasoning can be used to show that the conjectures are true in all circumstances. Inductive reasoning cannot prove propositions; valid conclusions and proof require deduction.

- Explain and illustrate the importance of generalization in mathematics and its relationship to inductive and deductive reasoning.

Example: No number of specific instances that illustrate the commutative property of addition can show that the property holds true for all real numbers, whereas $a + b = b + a$ (a and b real) is an axiom that includes all such cases.

c. Explain and illustrate the role of definitions, conjectures, theorems, proofs, and counterexamples in mathematical reasoning.

- Identify and give examples of definitions, conjectures, theorems, proofs, and counterexamples.
- Recognize flaws or gaps in the reasoning used to support an argument.
- Demonstrate through example or explanation how indirect reasoning can be used to establish a claim.

d. Make, test, and confirm or refute conjectures using a variety of methods.

- Use inductive reasoning to formulate conjectures and propose generalizations.
- Construct simple logical arguments and proofs; determine simple counterexamples.

D.C.3 Propositional logic

a. Use and interpret relational conjunctions (“and,” “or,” “not”), terms of causation (“if . . . then”) and equivalence (“if and only if”).

- Distinguish between the common uses of such terms in everyday language and their use in mathematics.
- Relate and apply these operations to situations involving sets.

b. Describe logical statements using terms such as assumption, hypothesis, conclusion, converse, and contrapositive.

c. Recognize and avoid flawed reasoning, including, but not limited to, “Since $A \Rightarrow B$, therefore $B \Rightarrow A$.”

d. Recognize syllogisms, tautologies and circular reasoning and use them to assess the validity of an argument.

D.E.1 Quantitative applications

a. Identify and apply the quantitative issues underlying voting, elections, and apportionment.

- Compare features of common methods of voting (e.g., majority, plurality, runoff) and describe how their results can vary.
- Identify, compare, and apply methods of apportionment.

Example: Devise a student government where the seats are fairly apportioned among all constituencies.

b. Know and use methods of fair division and negotiation strategies.

D.E.2 Sequences and series

- a. Know and use subscript notation to represent the general term of a sequence and summation notation to represent partial sums of a sequence.
- b. Derive and apply the formulas for the general term of arithmetic and geometric sequence.
- c. Derive and apply formulas to calculate sums of finite arithmetic and geometric series.
- d. Derive and apply formulas to calculate sums of infinite geometric series whose common ratio r is in the interval $(-1, 1)$.
- e. Model, analyze, and solve problems using sequences and series.

Examples: Determine the amount of interest paid over five years of a loan; determine the age of a skeleton using carbon dating; determine the cumulative relative frequency in an arithmetic or geometric growth situation.

D.E.3 Recursive equations

- a. Convert the recursive model for discrete linear growth (A_1 is given and $A_{n+1} = A_n + d$ for $n > 1$, d a constant difference) to a closed linear form ($A_n = a + d(n - 1)$).

This model generates an arithmetic sequence.

- b. Convert the recursive model of discrete population growth (P_1 is given and $P_{n+1} = rP_n$, for $n > 1$, r a constant growth rate) to a closed exponential form ($P_n = ar^{n-1}$).

This model generates a geometric sequence.

- c. Analyze, define, and calculate sequences that are neither arithmetic nor geometric using recursive methods.

It is often much clearer and less difficult to represent sequences recursively than in closed form.

D.E.4 Digital codes

- a. Interpret common digital codes (e.g., zip codes, universal product codes (UPCs) and ISBNs on books) and identify their special characteristics.
- b. Understand, evaluate, and compare how error detection and error correction are accomplished in different common codes.

Examples: Codes read by scanners; transmission of digital pictures over noisy channels; playing a scratched CD recording.

- Know the meaning of a *check digit* and how it is calculated.

- c. Identify characteristics of common forms of data compression (e.g., mp3, jpeg, and gif).

d. Analyze the concepts underlying public-key encryption and digital signatures that enable messages to be transmitted securely.

One method uses the fact that factoring a big number is much more difficult than creating one by multiplying two primes.

D.E.5 Mathematical induction

a. Analyze and describe how mathematical induction rests on the definition of whole numbers and explain how proof by mathematical induction establishes a proposition.

Mathematical induction is a unique logical principle used to establish the truth of an infinite sequence of statements that are indexed by positive integers, that is, true for all positive integers k . Formally, the principle states that if $p(1)$ is true, and if, for each integer k , $p(k + 1)$ is true whenever $p(k)$ is true, then $p(n)$ is true for all n .

b. Identify common theorems that can be proved by mathematical induction and explain why this method of proof works for these theorems.

c. Use mathematical induction to prove simple propositions.

Example: Prove that for every positive integer n , $1 + 2 + 3 + 4 + \dots + n = n(n + 1)/2$.

D.E.6 Proof by contradiction

a. Analyze and explain how proof by contradiction can be used to establish a proposition.

*Proof by contradiction is an indirect method of reasoning that shows that a conclusion cannot be false rather than showing directly that it is true. Typically, a proof by contradiction begins by assuming that the desired conclusion is not true and then uses correct reasoning to reach an absurd conclusion (such as that $1 = 0$). For this reason, the method is often known by its Latin name: *reductio ad absurdum*.*

b. Identify examples of theorems for which an indirect argument is useful and assess whether an indirect argument is useful to prove a particular theorem.

Example: Establishing that $\sqrt{2}$ is irrational can be done using an indirect argument.

c. Use an indirect argument to prove a result.

Examples: Any non-zero rational multiple of an irrational number is irrational; the sum of a rational number and an irrational number is irrational.

Algebra (A)

The language of algebra provides the means to express and illuminate mathematical relationships. Multiple representations—verbal, symbolic, numeric, and graphic—are used to describe change, to express the interaction of forces, and to describe and compare patterns. Algebra allows its users to generate new knowledge by drawing broad, rigorous generalizations from specific examples. Every mathematical strand makes extensive use of algebra to symbolize, to clarify, and to communicate its concepts and content. Learning algebra is an important step in a student’s cognitive mathematical development. It opens the door to organized abstract thinking, supplies a tool for logical reasoning, and helps us to model and understand the quantitative relationships so vital in today’s world.

A.A.1 Variables and expressions

a. Interpret and compare the different uses of variables and describe patterns, properties of numbers, formulas, and equations using variables.

- Compare the different uses of variables.

Examples: When $a + b = b + a$ is used to state the commutative property for addition, the variables a and b represent all real numbers; the variable a in the equation $3a - 7 = 8$ is a temporary placeholder for the one number, 5, that will make the equation true; the symbols C and r refer to specific attributes of a circle in the formula $C = 2\pi r$; the variable m in the slope-intercept form of the line, $y = mx + b$ serves as a parameter describing the slope of the line.

- Express patterns, properties, formulas, and equations using and defining variables appropriately for each case.

b. Analyze and identify characteristics of algebraic expressions.

- Analyze expressions to identify when an expression is the sum of two or more simpler expressions (called terms) or the product of two or more simpler expressions (called factors).
- Identify single-variable expressions as linear or non-linear.

c. Evaluate, interpret, and construct simple algebraic expressions.

- Evaluate a variety of algebraic expressions at specified values of their variables.

Algebraic expressions to be evaluated include polynomial and rational expressions as well as those involving radicals and absolute value.

Example: Evaluate $3x^2 - 2y^3 + \frac{\sqrt{w}}{|xy|}$ for $x = 6$, $y = -3$, and $w = 81$.

- Write linear and quadratic expressions representing quantities arising from geometric and real-world contexts.

Examples: Area of a rectangle of length l and width w ; area of a circle of radius r ; cost of buying 5 apples at price p and 7 oranges at price q .

- Analyze the structure of an algebraic expression and identify the resulting characteristics.

Example: $-5(u^2 + 4)$ is a product of two factors, the second of which is always positive because it is the sum of a square and a positive number; since the first factor is negative, the algebraic expression is negative for all values of u .

d. Identify and transform expressions into equivalent expressions.

Two algebraic expressions are equivalent if they yield the same result for every value of the variables in them.

- Use commutative, associative, and distributive properties of number operations to transform simple expressions into equivalent forms in order to collect like terms or to reveal or emphasize a particular characteristic.

Examples: Add, subtract, and multiply linear expressions, such as

$$\begin{aligned}(2x + 5) + (3 - 2x) &= 2x + 5 + 3 - 2x = 8, \\(2x + 5) - (3 - 2x) &= 2x + 5 - 3 + 2x = 2 + 4x, \text{ or} \\-5(3 - 2x) &= -15 + 10x.\end{aligned}$$

Transform simple nonlinear expressions, such as

$$\begin{aligned}(3p)(5q) &= 15pq \text{ or} \\n(n + 1) &= n^2 + n.\end{aligned}$$

- Rewrite linear expressions in the form $ax + b$ for constants a and b .
- Choose different but equivalent expressions for the same quantity that are useful in different contexts.

Example: $p + 0.07p$ shows the breakdown of the cost of an item into the price p and the tax of 7%, whereas $(1.07)p$ is a useful equivalent form for calculating the total cost.

e. Determine whether two algebraic expressions are equivalent.

- Demonstrate equivalence through algebraic transformations.
- Show that expressions are not equivalent by evaluating them at the same value(s) to get different results.
- Show that certain expressions are equivalent by checking at a small number of different values (e.g., two linear expressions are equivalent if they yield equal results at two distinct values of the variable), and identify the special circumstances under which this may be true.

Great care must be taken to demonstrate that, in general, a finite number of instances is not sufficient to demonstrate equivalence.

- Set each expression equal to y , consider all ordered pairs of these newly constructed equations, and know that if the graph of all ordered pairs that satisfy one equation is identical to the graph of all ordered pairs that satisfy the other, then the expressions are equivalent.

f. Apply the properties of exponents to transform variable expressions involving integral exponents.

- Know and apply the laws of exponents.

Examples: $a^p \cdot a^q = a^{p+q}$; $\frac{x^5}{x^7} = \frac{1}{x^2} = x^{-2}$; $9^x = 3^{2x}$; $64^{\frac{5}{6}} = 32$.

- Factor out common factors with exponents.

Factoring transforms an expression that was written as a sum or difference into one that is written as a product.

Examples: $6v^7 + 12v^5 - 8v^3 = 2v^3(3v^4 + 6v^2 - 4)$; $3x(x + 1)^2 - 2(x + 1)^2 = (x + 1)^2(3x - 2)$.

"Chunking" is a term often used to describe treating an expression, such as the $x + 1$ above, as a single entity.

g. Interpret rational exponents; translate between rational exponents and notation involving integral powers and roots.

Examples: $64^{\frac{5}{6}} = (64^{\frac{1}{6}})^5 = 2^5 = 32$; $(8b^6)^{\frac{1}{3}} = 2b^2$; $x^{\frac{4}{5}} = \sqrt[5]{x^4} = (\sqrt[5]{x})^4$.

A.A.2 Functions

a. Determine whether a relationship is or is not a function.

In general, a function is a rule that assigns a single element of one set—the output set—to each element of another set—the input set. The set of all possible inputs is called the domain of the function, while the set of all outputs is called the range.

- Identify the independent (input) and dependent (output) quantities/variables of a function.

b. Represent and interpret functions using graphs, tables, words and symbols.

- Make tables of inputs x and outputs $f(x)$ for a variety of rules that take numbers as inputs and produce numbers as outputs.

The notation $f(x)$ or $P(t)$ represents the number that the function f or P assigns to the input x or t .

- Define functions algebraically, e.g., $g(x) = 3 + 2(x - x^2)$.

When functions are defined by algebraic expressions, these expressions are sometimes called formulas. Not every function can be defined by means of an algebraic expression. Many are stated using algorithms or verbal descriptions.

Spreadsheet software packages offer an abundant source of function rules.

- Create the graph of a function f by plotting the ordered pairs $(x, f(x))$ in the coordinate plane.
- Analyze and describe the behavior of a variety of simple functions using tables, graphs, and algebraic expressions.

Examples: $f(x) = 3x + 1$; $f(x) = 3x$; $f(x) = 3x^2 + 1$; $f(x) = 2x^3$; $f(x) = 2^x$; $f(x) = 3/x$.

To understand the breadth of the function concept, it is important for students to work with a variety of examples.

- Construct and interpret functions that describe simple problem situations using expressions, graphs, tables, and verbal descriptions and move flexibly among these multiple representations.

Caution should be taken when using tables, since they only indicate the value of the function at a finite number of points and could arise from many different functions.

A.A.3 Linear functions

a. Analyze and identify linear functions of one variable.

A function exhibiting a constant rate of change is called a linear function. A constant rate of change means that for any pair of inputs x_1 and x_2 , the ratio of the corresponding change in value $f(x_2) - f(x_1)$ to the change in input $x_2 - x_1$ is constant (i.e., it does not depend on the inputs).

- Explain why any function defined by a linear algebraic expression has a constant rate of change.

Examples: $f(x) = 2x$; $f(x) = 5 - 3x$; $f(\text{side of square}) = \text{perimeter of square}$.

- Explain why the graph of a linear function defined for all real numbers is a straight line, and identify its constant rate of change and create the graph.
- Explain why a vertical line is not the graph of a function.
- Determine whether the rate of change of a specific function is constant; use this to distinguish between linear and nonlinear functions.

b. Know the definitions of x - and y -intercepts, know how to find them, and use them to solve problems.

An x -intercept is the value of x where $f(x) = 0$. A y -intercept is the value of $f(0)$.

c. Know the definition of slope, calculate it, and use slope to solve problems.

The slope of a linear function is its constant rate of change.

- Know that a line with positive slope tilts from lower left to upper right, whereas a line with a negative slope tilts from upper left to lower right.
- Know that a line with slope equal to zero is horizontal, while the slope of a vertical line is undefined.

d. Express a linear function in several different forms for different purposes.

- Recognize that in the form $f(x) = mx + b$, m is the slope, or constant rate of change of the graph of f , that b is the y -intercept, and that in many applications of linear functions, b defines the initial state of a situation; express a function in this form when this information is given or needed.

- Recognize that in the form $f(x) = m(x - x_0) + y_0$, the graph of $f(x)$ passes through the point (x_0, y_0) ; express a function in this form when this information is given or needed.

e. Recognize contexts in which linear models are appropriate; determine and interpret linear models that describe linear phenomena.

Common examples of linear phenomena include distance traveled over time for objects traveling at constant speed; shipping costs under constant incremental cost per pound; conversion of measurement units (e.g., pounds to kilograms or degrees Celsius to degrees Fahrenheit); cost of gas in relation to gallons used; the height and weight of a stack of identical chairs.

- Identify situations that are linear and those that are not linear and justify the categorization based on whether the rate of change is constant or varies.
- Express a linear situation in terms of a linear function $f(x) = mx + b$ and interpret the slope (m) and the y -intercept (b) in terms of the original linear context.

A.A.4 Proportional functions

a. Recognize, graph, and use direct proportional relationships.

A proportion is composed of two pairs of real numbers, (a, b) and (c, d) , with at least one member of each pair non-zero, such that both pairs represent the same ratio. A linear function in which $f(0) = 0$ represents a direct proportional relationship. The function $f(x) = kx$, where k is constant describes a direct proportional relationship.

- Show that the graph of a direct proportional relationship is a line that passes through the origin $(0, 0)$ whose slope is the constant of proportionality.
- Compare and contrast the graphs of $x = k$, $y = k$, and $y = kx$, where k is a constant.
- If $f(x)$ is a linear function, show that $g(x) = f(x) - f(0)$ represents a direct proportional relationship.

In this case, $g(0) = 0$, so $g(x) = kx$. The graph of $f(x) = mx + b$ is the graph of the direct proportional relationship $g(x) = mx$ shifted up (or down) by b units. Since the graph of $g(x)$ is a straight line, so is the graph of $f(x)$.

b. Recognize, graph, and use reciprocal relationships.

A function of the form $f(x) = k/x$ where k is constant describes a reciprocal relationship. The term "inversely proportional" is sometimes used to identify such relationships, however, this term can be very confusing since the word "inverse" is also used in the term "inverse function" (the function $f^{-1}(x)$ with the property that $f \text{ of } f^{-1}(x) = f^{-1} \text{ of } f(x) = x$, the identity function).

- Analyze the graph of $f(x) = k/x$ and identify its key characteristics.

The graph of $f(x) = k/x$ is not a straight line and does not cross either the x - or the y -axis (i.e., there is no value of x for which $f(x) = 0$, nor is there any value for $f(x)$ if $x = 0$). It is a curve consisting of two disconnected branches, called a hyperbola.

- Recognize quantities that are inversely proportional and express their relationship symbolically.

Example: The relationship between lengths of the base and side of a rectangle with fixed area.

c. Distinguish among and apply linear, direct proportional, and reciprocal relationships.

- Identify whether a table, graph, formula, or context suggests a linear, direct proportional or reciprocal relationship.
- Create graphs of linear, direct proportional, and reciprocal functions by hand and using technology.
- Identify practical situations that can be represented by linear, direct, or inversely proportional relationships; analyze and use the characteristics of these relationships to answer questions about the situation.

d. Explain and illustrate the effect of varying the parameters m and b in the family of linear functions and varying the parameter k in the families of directly proportional and reciprocal functions.

A.A.5 Equations and identities

a. Distinguish among an equation, an expression, and a function.

An equation is a statement of equality between algebraic expressions or functions.

Example: If $f(x) = 3x + 2$ and $g(x) = 5x - 8$, the statement $f(x) = g(x)$ is an equation in one variable.

- Know that solving an equation means finding all its solutions.

A solution of an equation (in one variable) is a value of the variable that makes the equation true. Because the solutions of an equation are often not known (or at least not apparent from the form of the equation), a variable in an equation is often called an unknown.

- Predict the number of solutions that should be expected for various simple equations and identities.
- Explain why solutions to the equation $f(x) = g(x)$ are the x -values (abscissas) of the set of points in the intersection of the graphs of the functions $f(x)$ and $g(x)$.
- Recognize that $f(x) = 0$ is a special case of the equation $f(x) = g(x)$ and solve the equation $f(x) = 0$ by finding all values of x for which $f(x) = 0$.

The solutions to the equation $f(x) = 0$ are called roots of the equation or zeros of the function. They are the values of x where the graph of the function f crosses the x -axis. In the special case where $f(x)$ equals 0 for all values of x , $f(x) = 0$ represents a constant function where all elements of the domain are zeros of the function.

Example: The graph of the linear function $f(x) = 2x - 4$ crosses the x -axis at $x = 2$. Hence, 2 is a root of the equation $f(x) = 0$, since $f(2) = 2(2) - 4 = 0$.

Beware of the confusion inherent in two apparently different meanings of the word "root": the root of an equation (e.g., $3x^2 - 4x + 1 = 0$) and the root of a number (e.g., $\sqrt{5}$). Although different, these uses do arise from a common source: The root of a number such as $\sqrt{5}$ is a root (or a solution) of an associated equation, namely $x^2 - 5 = 0$.

- Interpret the notation for the equation $y = f(x)$ as a function for which each specific input, x , has a specific y -value as its output.

In this representation, y stands for the output $f(x)$ of the function f and corresponds to the y -axis on an x - y coordinate grid.

Example: The two-variable equation $y = 3x + 8$ corresponds to the single-variable linear function $f(x) = 3x + 8$.

b. Solve linear and simple nonlinear equations involving several variables for one variable in terms of the others.

Example: Solve $A = \pi r^2 h$ for h or for r .

c. Interpret identities as a special type of equation and identify their key characteristics.

An identity is an equation for which all values of the variables are solutions. Although an identity is a special type of equation, there is a difference in practice between the methods for solving equations that have a small number of solutions and methods for proving identities. For example, $(x+2)^2 = x^2 + 4x + 4$ is an identity which can be proved by using the distributive property, whereas $(x+2)^2 = x^2 + 3x + 4$ is an equation that can be solved by collecting all terms on one side.

- Use identities to transform expressions.

d. Make regular fluent use of basic algebraic identities such as $(a + b)^2 = a^2 + 2ab + b^2$; $(a - b)^2 = a^2 - 2ab + b^2$; and $(a + b)(a - b) = a^2 - b^2$.

- Use the distributive law to derive each of these formulas.

Examples: $(a + b)(a - b) = (a + b)a - (a + b)b = (a^2 + ab) - (ab + b^2) = a^2 + ab - ab - b^2 = a^2 - b^2$; applying this to specific numbers, $37 \cdot 43 = (40 - 3)(40 + 3) = 1,600 - 9 = 1,591$.

- Use geometric constructs to illustrate these formulas.

Example: Use a partitioned square or tiles to provide a geometric representation of $(a + b)^2 = a^2 + 2ab + b^2$.

e. Create, interpret, and apply mathematical models to solve problems arising from contextual situations.

Mathematical modeling consists of recognizing and clarifying mathematical structures that are embedded in other contexts, formulating a problem in mathematical terms, using mathematical strategies to reach a solution, and interpreting the solution in the context of the original problem.

- Distinguish relevant from irrelevant information, identify missing information, and find what is needed or make appropriate estimates.

- Apply problem solving heuristics to practical problems: Represent and analyze the situation using symbols, graphs, tables, or diagrams; assess special cases; consider analogous situations; evaluate progress; check the reasonableness of results; and devise independent ways of verifying results.

A.A.6 Linear equations and inequalities

a. Solve linear equations in one variable algebraically.

An equation of the form $f(x) = g(x)$ is linear if the function $f(x) - g(x)$ is linear. Combining terms makes each linear equation in a single variable equivalent to an equation in the standard form $ax + b = 0$.

- Solve equations using the facts that equals added to equals are equal and that equals multiplied by equals are equal. More formally, if $A = B$ and $C = D$, then $A + C = B + D$ and $AC = BD$.

Together with the ordinary laws of arithmetic (commutative, associative, distributive), these principles justify the steps used to transform linear equations into equivalent equations in standard form and then solve them.

- Using the fact that a linear expression $ax + b$ is formed using the operations of multiplication by a constant followed by addition, solve an equation $ax + b = 0$ by reversing these steps.
- Be alert to anomalies caused by dividing by 0 (which is undefined), or by multiplying both sides by 0 (which will produce equality even when things were originally unequal).

Example: Multiplying both sides of an equation by $x - 1$ is appropriate only when $x \neq 1$.

b. Solve and graph the solution of linear inequalities in one variable.

A solution to a linear inequality in one variable consists of all points on the number line whose coordinates satisfy the inequality.

- Graph a linear inequality in one variable and explain why the graph is always a half-line (open or closed).
- Know that the solution set of a linear inequality in one variable is infinite, and contrast this with the solution set of a linear equation in one variable.

It is also possible that some contextual situations may limit the reasonable solutions of a linear inequality to a finite number.

- Explain why, when both sides of an inequality are multiplied or divided by a negative number, the direction of the inequality is reversed, but that when all other basic operations involving non-zero numbers are applied to both sides, the direction of the inequality is preserved.

c. Identify the relationship between linear functions of one variable and linear equations in two variables.

- Translate fluently between the linear function of one variable $f(x) = mx + b$ and the related linear equation in two variables $y = mx + b$.
- Rewrite a linear equation in two variables in any of three forms: $ax + by = c$, $ax + by + c = 0$, or $y = mx + b$; select a form depending upon how the equation is to be used.
- Know that the graph of a linear equation in two variables consists of all points (x, y) in the coordinate plane that satisfy the equation and explain why, when x can be any real number, such graphs are straight lines.

d. Use graphs to help solve linear equations in one variable.

- Explain why the solution to an equation in standard (or polynomial) form ($ax + b = 0$) will be the point at which the graph of $f(x) = ax + b$ crosses the x -axis.
- Identify the solution of an equation that is in the form $f(x) = g(x)$ and relate the solution to the x -value (*abscissa*) of the point at which the graphs of the functions $f(x)$ and $g(x)$ intersect.

Example: To solve the linear equation $3x + 1 = x + 5$, graph $f(x) = 3x + 1$ and $g(x) = x + 5$. The graphs of $f(x)$ and $g(x)$ intersect at the point $(2, 7)$; thus the solution to the linear equation is $x = 2$. Alternatively, the linear equation $3x + 1 = x + 5$ is equivalent to $2x - 4 = 0$. This yields a single linear function $h(x) = 2x - 4$ whose graph crosses the x -axis at $x = 2$.

e. Represent any straight line in the coordinate plane by a linear equation in two variables.

- Represent any line in its standard form $ax + by = c$ whether or not the line is the graph of a function.
Vertical lines have the equation $x = k$ (or $1x + 0y = k$) with undefined slope and do not represent functions.
- Know that pairs of non-vertical lines have the same slope **only** if they are parallel (or the same line) and slopes that are negative reciprocals **only** if they are perpendicular; apply these relationships to analyze and represent equations.

f. Solve and graph the solution of a linear inequality in two variables.

- Know what it means to be a solution of a linear inequality in two variables, represent solutions algebraically and graphically, and provide examples of ordered pairs that lie in the solution set.
- Graph a linear inequality in two variables and explain why the graph is always a half-plane (open or closed).
In analogy with the vocabulary of equations, the collection of all points (x, y) that satisfy the linear inequality $ax + by < c$ is called the graph of the inequality. These points lie entirely in one of the half-planes determined by the graph of the equation $ax + by = c$.

g. Recognize and solve problems that can be modeled using linear equations or inequalities in one or two variables; interpret the solution(s) in terms of the context of the problem.

Common problems are those that involve time/rate/distance, percentage increase or decrease, ratio and proportion.

- Represent linear relationships using tables, graphs, verbal statements, and symbolic forms; translate among these forms to extract information about the relationship.

h. Solve equations and inequalities involving the absolute value of a linear expression in one variable.

A.B.1 Quadratic functions

a. Identify quadratic functions expressed in multiple forms; identify the specific information each form clarifies.

- Express a quadratic function as a polynomial, $f(x) = ax^2 + bx + c$, where a , b , and c are constants with $a \neq 0$, and identify its graph as a parabola that opens up when $a > 0$ and down when $a < 0$; relate c to where the graph of the function crosses the y -axis.
- Express a quadratic function in factored form, $f(x) = (x - r)(x - s)$, when r and s are integers; relate the factors to the solutions of the equation $(x - r)(x - s) = 0$ ($x = r$ and $x = s$) and to the points $((r, 0)$ and $(s, 0))$ where the graph of the function crosses the x -axis.

b. Graph quadratic functions and use the graph to help locate zeros.

A zero of a quadratic function $f(x) = ax^2 + bx + c$ is a value of x for which $f(x) = 0$.

- Sketch graphs of quadratic functions using both graphing calculators and tables of values.
- Estimate the real zeros of a quadratic function from its graph.
- Identify quadratic functions that do not have real zeros by the behavior of their graphs.

A quadratic function that does not cross the horizontal axis has no real zeros.

c. Recognize contexts in which quadratic models are appropriate; determine and interpret quadratic models that describe quadratic phenomena.

Examples: The relationship between length of the side of a square and its area; the relationship between time and distance traveled for a falling object.

A.B.2 Simple quadratic equations

a. Solve quadratic equations that can be easily transformed into the form $(x - a)(x - b) = 0$ or $(x + a)^2 = b$, for a and b integers.

b. Estimate the roots of a quadratic equation from the graph of the corresponding function.

c. Solve simple quadratic equations that arise in the context of practical problems and interpret their solutions in terms of the context.

Examples: Determine the height of an object above the ground t seconds after it has been thrown upward at an initial velocity of v_0 feet per second from a platform d feet above the ground; find the area of a rectangle with perimeter 120 in terms of the length, L , of one side.

A.B.3 Systems of linear equations and inequalities

a. Solve systems of linear equations in two variables using algebraic procedures.

A system of simultaneous linear equations in two variables consists of two or more different linear equations in two variables. A solution to such a system is the set of ordered pairs of values (x_0, y_0) that makes all of the equations true.

- Determine whether a system of two linear equations has one solution, no solutions, or infinitely many solutions, and know that these are the only possibilities.

b. Use graphs to help solve systems of simultaneous linear equations in two variables.

- Use the graph of a system of equations in two variables to suggest solution(s).

Since the solution is a set of ordered pairs that satisfy the equations, it follows that these ordered pairs must lie on the graph of each of the equations in the system; the point(s) of intersection of the graphs is (are) the solution(s) to the system of equations.

Example: To solve the system $3x + 5y = 11$; $7x - 9y = 5$, first graph each of the two equations. It appears that the two graphs intersect at point $x_0 = 2, y_0 = 1$. Substitution of these values in both equations establishes that $(2, 1)$ is indeed a solution of both equations and the actual point of intersection.

- Represent the graphs of a system of two linear equations as two intersecting lines when there is one solution, parallel lines when there is no solution, and the same line when there are infinitely many solutions.

c. Solve systems of two or more linear inequalities in two variables and graph the solution set.

Example: The set of points (x, y) that satisfy all three inequalities $5x - y \geq 3$, $3x + y \leq 10$, and $4x - 3y \leq 6$ is a triangle, the intersection of three half-planes whose points satisfies each inequality separately.

d. Solve systems of simultaneous linear equations in three variables using algebraic procedures.

A system of simultaneous linear equations in three variables consists of three or more different linear equations in three variables. A solution to such a system is the set of ordered triples of values (x_0, y_0, z_0) that makes all of the equations true.

e. Describe the possible arrangements of the graphs of three linear equations in three variables and relate these to the number of solutions of the corresponding system of equations.

f. Recognize and solve problems that can be modeled using a system of linear equations or inequalities; interpret the solution(s) in terms of the context of the problem.

Examples: Break-even problems, such as those comparing costs of two services; optimization problems that can be approached through linear programming.

A.C.1 Elementary functions

a. Identify key characteristics of absolute value, step, and other piecewise-linear functions and graph them.

- Interpret the algebraic representation of a piecewise-linear function; graph it over the appropriate domain.
- Write an algebraic representation for a given piecewise-linear function.
- Determine vertex, slope of each branch, intercepts, and end behavior of an absolute value graph.
- Recognize and solve problems that can be modeled using absolute value, step, and other piecewise-linear functions.

Examples: Postage rates, cellular telephone charges, tax rates.

b. Graph and analyze exponential functions and identify their key characteristics.

- Know that exponential functions have the general form $f(x) = ab^x + c$ for $b > 0$, $b \neq 1$; identify the general shape of the graph and its lower or upper limit (asymptote).
- Explain and illustrate the effect that a change in a parameter has on an exponential function (a change in a , b , or c for $f(x) = ab^x + c$).

c. Analyze power functions and identify their key characteristics.

Power functions include positive integer power functions such as $f(x) = -3x^4$, root functions such as $f(x) = 5\sqrt{x}$ and $f(x) = 4x^{\frac{1}{3}}$ and reciprocal functions such as $f(x) = kx^{-4}$.

- Recognize that the inverse proportional function $f(x) = k/x$ ($f(x) = kx^n$ for $n = -1$) and the direct proportional function $f(x) = kx$ ($f(x) = kx^n$ for $n = 1$) are special cases of power functions.
- Distinguish between *odd* and *even* power functions.

Examples: When the exponent of a power function is a positive integer, then even power functions have either a minimum or maximum value, while odd power functions have neither; even power functions have reflective symmetry over the y -axis, while odd power functions demonstrate rotational symmetry about the origin.

d. Transform the algebraic expression of power functions using properties of exponents and roots.

Example: $f(x) = 3x^2 \left(-2x^{-\frac{3}{2}} \right)$ can be more easily identified as a root function once it is

rewritten as $f(x) = -6x^{\frac{1}{2}} = -6\sqrt{x}$.

- Explain and illustrate the effect that a change in a parameter has on a power function (a change in a or n for $f(x) = ax^n$).

e. Distinguish among the graphs of simple exponential and power functions by their key characteristics.

Be aware that it can be very difficult to distinguish graphs of these various types of functions over small regions or particular subsets of their domains. Sometimes the context of an underlying situation can suggest a likely type of function model.

- Decide whether a given exponential or power function is suggested by the graph, table of values, or underlying context of a problem.
- Distinguish between the graphs of exponential growth functions and those representing exponential decay.
- Distinguish among the graphs of power functions having positive integral exponents, negative integral exponents, and exponents that are positive unit fractions ($f(x) = x^{\frac{1}{n}}$, $n \geq 0$).
- Identify and explain the symmetry of an even or odd power function.
- Where possible, determine the domain, range, intercepts, asymptotes, and end behavior of exponential and power functions.

Range is not always possible to determine with precision.

f. Recognize and solve problems that can be modeled using exponential and power functions; interpret the solution(s) in terms of the context of the problem.

- Use exponential functions to represent *growth* functions, such as $f(x) = an^x$ ($a > 0$ and $n > 1$), and *decay* functions, such as $f(x) = an^{-x}$ ($a > 0$ and $n > 1$).

Exponential functions model situations where change is proportional to quantity (e.g., compound interest, population growth, radioactive decay).

- Use power functions to represent quantities arising from geometric contexts such as length, area, and volume.

Examples: The relationships between the radius and area of a circle, between the radius and volume of a sphere, and between the volumes of simple three-dimensional solids and their linear dimensions.

- Use the laws of exponents to determine exact solutions for problems involving exponential or power functions where possible; otherwise approximate the solutions graphically or numerically.

g. Explain, illustrate, and identify the effect of simple coordinate transformations on the graph of a function.

- Interpret the graph of $y = f(x - a)$ as the graph of $y = f(x)$ shifted $|a|$ units to the right ($a > 0$) or the left ($a < 0$).
- Interpret the graph of $y = f(x) + a$ as the graph of $y = f(x)$ shifted $|a|$ units up ($a > 0$) or down ($a < 0$).
- Interpret the graph of $y = f(ax)$ as the graph of $y = f(x)$ expanded horizontally by a factor of $\frac{1}{|a|}$ if $0 < |a| < 1$ or compressed horizontally by a factor of $|a|$ if $|a| > 1$ and reflected over the y -axis if $a < 0$.
- Interpret the graph of $y = af(x)$ as the graph of $y = f(x)$ compressed vertically by a factor of $\frac{1}{|a|}$ if $0 < |a| < 1$ or expanded vertically by a factor of $|a|$ if $|a| > 1$ and reflected over the x -axis if $a < 0$.

A.C.2 Polynomial functions

a. Transform quadratic functions and interpret their graphical forms.

- Write a quadratic function in polynomial or standard form, $f(x) = ax^2 + bx + c$, to identify the y -intercept of the function's parabolic graph or the x -coordinate of its vertex, $x = -\frac{b}{2a}$.
- Write a quadratic function in factored form, $f(x) = a(x - b)(x - c)$, to identify the zeros of the function's parabolic graph.
- Write a quadratic function in vertex form, $f(x) = a(x - h)^2 + k$, to identify the vertex and axis of symmetry of the function.
- Describe the effect that changes in the leading coefficient or constant term of $f(x) = ax^2 + bx + c$ have on the shape, position, and characteristics of the graph of $f(x)$.
Examples: If a and c have opposite signs, then the zeros of the quadratic function must be real and have opposite signs; varying c varies the y -intercept of the graph of the parabola; if a is positive, the parabola opens up, if a is negative, it opens down; as $|a|$ increases, the graph of the parabola is stretched vertically, i.e., it looks narrower.
- Determine domain and range, intercepts, axis of symmetry, and maximum or minimum for quadratic functions whose intercepts and vertices are real.

b. Analyze polynomial functions and identify their key characteristics.

- Know that polynomial functions of degree n have the general form $f(x) = ax^n + bx^{n-1} + \dots + px^2 + qx + r$ for n an integer, $n \geq 0$ and $a \neq 0$.

The degree of the polynomial function is the largest power of its terms for which the coefficient is non-zero.

- Know that a power function with an exponent that is a positive integer is a particular type of polynomial function, called a *monomial*, whose graph contains the origin.
- Distinguish among polynomial functions of low degree, i.e., constant functions, linear functions, quadratic functions, or cubic functions.
- Explain why every polynomial function of odd degree has at least one zero.
- Communicate understanding of the concept of the multiplicity of a root of a polynomial equation and its relationship to the graph of the related polynomial function.

If a zero, r_1 , of a polynomial function has multiplicity 3, $(x - r_1)^3$ is a factor of the polynomial. The graph of the polynomial touches the horizontal axis at r_1 but does not change sign (does not cross the axis) if the multiplicity of r_1 is even; it changes sign (crosses over the axis) if the multiplicity is odd.

c. Use key characteristics to identify the graphs of simple polynomial functions.

Simple polynomial functions include constant functions, linear functions, quadratic functions such as $f(x) = ax^2 + b$ or $g(x) = (x - a)(x + b)$, or cubic functions such as $f(x) = x^3$, $f(x) = x^3 - a$, or $f(x) = x(x - a)(x + b)$.

- Decide if a given graph or table of values suggests a simple polynomial function.
- Distinguish between the graphs of simple polynomial functions.
- Where possible, determine the domain, range, intercepts, and end behavior of polynomial functions.

It is not always possible to determine exact horizontal intercepts.

d. Recognize and solve problems that can be modeled using simple polynomial functions; interpret the solution(s) in terms of the context of the problem.

- Use polynomial functions to represent quantities arising from numeric or geometric contexts such as length, area, and volume.

Examples: The number of diagonals of a polygon as a function of the number of sides; the areas of simple plane figures as functions of their linear dimensions; the surface areas of simple three-dimensional solids as functions of their linear dimensions; the sum of the first n integers as a function of n .

- Solve simple polynomial equations and use technology to approximate solutions for more complex polynomial equations.

A.C.3 Polynomial and rational expressions and equations

a. Solve and graph quadratic equations having real solutions using a variety of methods.

- Solve quadratic equations having real solutions by factoring, by completing the square, and by using the quadratic formula.
- Use a calculator to approximate the roots of a quadratic equation and as an aid in graphing.
- Select and explain a method of solution (e.g., exact vs. approximate) that is effective and appropriate to a given problem.

b. Relate the coefficients a , b and c of the quadratic equation $ax^2 + bx + c = 0$ to its roots.

- Describe how the *discriminant* $D = b^2 - 4ac$ indicates the nature of the roots of the equation $ax^2 + bx + c = 0$.
The roots are real and distinct if $D > 0$, real and equal if $D = 0$, and not real if $D < 0$.
- Use the quadratic formula to prove that the abscissa of the vertex of the corresponding parabola is halfway between the roots of the equation.

c. Distinguish among linear, exponential, polynomial, rational and power expressions, equations, and functions by their symbolic form.

- Use the position of the variable in a term to determine the classification of the related expression, equation, or function.
Examples: $f(x) = 3^x$ is an exponential function because the variable is in the exponent, while $f(x) = x^3$ has the variable in the position of a base and is a power function; $f(x) = x^3 - 5$ is a polynomial function but not a power function because of the added constant.

d. Perform operations on polynomial expressions.

In general, a polynomial expression is any expression equivalent to one of the form $ax^n + bx^{n-1} + \dots + px^2 + qx + r$ for n an integer, $n \geq 0$ and $a \neq 0$. A polynomial expression is a sum of monomials.

- Add, subtract, multiply, and factor polynomials.
- Divide one polynomial by a lower-degree polynomial.

e. Know and use the binomial expansion theorem.

- Relate the expansion of $(a + b)^n$ to the possible outcomes of a binomial experiment and the n^{th} row of Pascal's triangle.

f. Use factoring to reduce rational expressions that consist of the quotient of two simple polynomials.

g. Perform operations on simple rational expressions.

Simple rational expressions are those whose denominators are linear or quadratic polynomial expressions.

- Add, subtract, multiply, and divide rational expressions having monomial or binomial denominators.

- Rewrite complex fractions composed of simple rational expressions as a simple fraction in lowest terms.

$$\text{Example: } \frac{\left(\frac{a+b}{\frac{1}{a} + \frac{1}{b}}\right)}{\frac{ab}{ab}} = \frac{(a+b)}{\frac{(b+a)}{ab}} = (a+b) \cdot \frac{ab}{(b+a)} = ab.$$

A.D.1 General quadratic equations and inequalities

a. Solve and graph quadratic equations having complex solutions.

- Use the quadratic formula to solve any quadratic equation and write it as a product of linear factors.
- Use the discriminant $D = b^2 - 4ac$, to determine the nature of the roots of the equation $ax^2 + bx + c = 0$.
- Show that complex roots of a quadratic equation having real coefficients occur in conjugate pairs.

b. Solve and graph quadratic inequalities in one or two variables.

Example: Solve $(x - 5)(x + 1) > 0$ and relate the solution to the graph of $(x - 5)(x + 1) > y$.

c. Manipulate simple quadratic equations to extract information.

Example: Use the completing the square method to determine the center and radius of a circle from its equation given in general form.

A.D.2 Rational and radical equations and functions

a. Solve simple rational and radical equations in one variable.

- Use algebraic, numerical, graphical, and/or technological means to solve rational equations.
- Use algebraic, numerical, graphical, and/or technological means to solve equations involving a radical.
- Know which operations on an equation produce an equation with the same solutions and which may produce an equation with fewer or more solutions (lost or extraneous roots) and adjust solution methods accordingly.

b. Graph simple rational and radical functions in two variables.

- Graph rational functions in two variables; identify the domain, range, intercepts, zeros, and asymptotes of the graph.
- Graph simple radical functions in two variables; identify the domain, range, intercepts, and zeros of the graph.

- Relate the algebraic properties of a rational or radical function to the geometric properties of its graph.

Examples: The graph of $f(x) = \frac{x-2}{x^2-1}$ has vertical asymptotes at $x = 1$ and $x = -1$,

while the graph of $f(x) = \frac{x-2}{x^2-4}$ has a vertical asymptote at $x = -2$ but a hole at $(2,$

$\frac{1}{4}$); the graph of $f(x) = \sqrt{x+5} - 2$ is the same as the graph of $f(x) = \sqrt{x}$ translated five units to the left and 2 units down.

A.E.1 Trigonometric functions

a. Relate sine and cosine functions to a central angle of the unit circle.

- Interpret the sine, cosine, and tangent functions corresponding to a central angle of the unit circle in terms of horizontal and vertical sides of right triangles based on that central angle.

b. Define and graph trigonometric functions over the real numbers.

- Use the unit circle to extend the domain of the sine and cosine function to the set of real numbers.
- Explain and use *radian measures* for angles; convert between radian and degree measure.
- Create, interpret, and identify key characteristics (period, amplitude, vertical shift, phase shift) of basic trigonometric graphs (sine, cosine, tangent).
- Identify the zeros of trigonometric functions that have a vertical shift of 0.
- Describe the effect that changes in each of the coefficients of $f(x) = A \sin B(x - C) + D$ have on the position and key characteristics of the graph of $f(x)$.

c. Analyze periodic functions and identify their key characteristics.

Periodic functions are used to describe cyclic behaviors.

- Recognize periodic phenomena.
Examples: The height above the pavement of a point on the tread of a truck tire; the length of time from sunrise to sunset over a 10-year period.
- Identify the length of a cycle in situations exhibiting periodic behavior and use it to make predictions.
- Use basic properties of frequency and amplitude to solve problems.
- Recognize that the graphs of trigonometric functions represent periodic behavior.

d. Recognize and solve problems that can be modeled using equations and inequalities involving trigonometric functions.

Examples: Circular motion as in approximate orbits, water wheels, or Ferris wheels; periodic behavior as in sound waves, tides, or minutes of daylight.

- Demonstrate graphically the relation between the sine function and common examples of harmonic motion.

e. Derive and use basic trigonometric identities.

- Derive the basic Pythagorean identities for sine and cosine, for tangent and secant, and for cotangent and cosecant.
- Know and use the angle addition formulas for sine, cosine, and tangent.
- Derive and use formulas for sine, cosine, and tangent of double angles.

f. Solve geometric problems using the sine and cosine functions.

- Know and use the Law of Sines and the Law of Cosines to solve problems involving triangles.
- Know and use the area formula $\text{Area} = ab \sin(C)$ to determine the area of triangle ABC.

A.E.2 Matrices and linear equations

a. Know and use matrix notation for rows, columns, and entries of cells.

b. Compute the determinant of a 2x2 or 3x3 matrix.

c. Know and perform addition, subtraction, and scalar multiplication of matrices.

- Recognize that matrix addition is associative and commutative and explain why that is the case.
- Distinguish between multiplication of a matrix by a scalar (a number or variable representing a number) and the multiplication of two matrices.

d. Know and perform matrix multiplication.

- Describe the characteristics of matrices that can be multiplied and those that cannot.
- Utilize knowledge of the algorithm for matrix multiplication. Compute the product by hand for 2×2 or 3×3 matrices and use technology for matrices of larger dimension.
- Know that matrix multiplication is not commutative and provide examples of square matrices A and B such that $AB \neq BA$.
- Know and apply the associative property of matrix multiplication.

The associative property of matrix multiplication states that if there are three matrices, A , B , and C such that AB and BC are defined, then $(AB)C$ and $A(BC)$ are defined and $(AB)C = A(BC)$.

e. Relate vector and matrix operations to transformations in the coordinate plane.

- Interpret vector and matrix addition as translation in the coordinate plane.
Examples: If vector $\vec{v} = (x, y)$, then $\vec{v} + (-1, 6)$ represents the translation of the plane 1 unit to the left and 6 units up; $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ is a way of representing the translation of the point $A(3, 1)$ to the left 1 unit and up 6 units; combining $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ describes the translation of the triangle $A(3,1)$, $B(-2,0)$, $C(4,5)$ to the left 1 unit and up 6 units to the new triangle $A'(2,7)$, $B'(-3,6)$, $C'(3,11)$, which is congruent to triangle ABC .
- Interpret matrix multiplication as reflection about the axes or rotation about the origin in the coordinate plane.
Examples: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ represents the rotation of the point $A(3, 1)$ 180° about the origin; $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ represents the reflection of the point $A(3, 1)$ over the y -axis.

f. Apply the concept of inverse to matrix multiplication.

- Know the definition and properties of the identity matrix.
- Find the inverse of a 2×2 matrix if the inverse exists.
- Use row reduction to find inverses of 3×3 matrices when the inverses exist.
- Use the inverse of a matrix, when one exists, to solve a matrix equation.

g. Write and solve systems of 2×2 and 3×3 linear equations in matrix form.

- Switch between equation notation and matrix notation for linear systems.
- Solve linear systems by row reduction.
- Solve linear systems using the inverse matrix.

If the inverse of a matrix associated with a linear system involving the same number of distinct equations as variables does not exist, then it is not always possible to solve the system; when it is possible, there will be infinitely many solutions.

A.E.3 Operations on functions

a. Compare and contrast properties of different types (families) of functions.

These types (families) include algebraic (linear, quadratic, polynomial, rational), piecewise (absolute value, step, piecewise-linear), and transcendental (trigonometric, exponential, logarithmic).

- Determine key characteristics of a function (e.g., domain, range, zeros, symmetries, asymptotes, end behavior) from its context or from its symbolic or graphical form.
Some details (e.g., range) may require technological assistance to determine.

- Recognize and solve problems that can be modeled with various types (families) of functions.

b. Analyze the transformations of a function from its graph, formula, or verbal description.

- Select a prototypical representation for each family of functions.
Examples: $f(x) = x^2$ is a prototype for quadratic functions; $g(x) = \sin(x)$ is a prototypical trigonometric function.
- Analyze a graph to identify properties that provide useful information about a given context.
- Identify changes in the graph of a function related to various transformations (vertical/horizontal translations, reflections over the x - or y -axis, dilation/contraction) and relate them to changes in the function's algebraic representation.
Example: The graph of $f(x) = -3x^2 + 4$ is a vertical dilation by a factor of 3 of the prototype $f(x) = x^2$ followed by a reflection over the x -axis and a translation 4 units up. The resulting vertex of the parabola $(0, 4)$ reflects these transformations and is evident when $f(x) = -3x^2 + 4$ is compared to the vertex form of a parabola $f(x) = a(x - h)^2 + k$.

c. Compute the sum, difference, product, and quotient of two functions.

d. Determine the composition of simple functions, including any necessary restrictions on the domain.

- Know the relationships among the identity function, composition of functions, and the inverse of a function, along with implications for the domain.

A.E.4 Inverse functions

a. Analyze characteristics of inverse functions.

- Identify the conditions under which the inverse of a function is a function.
- Determine whether two given functions are inverses of each other.
- Explain why the graph of a function and its inverse are reflections of one another over the line $y = x$.

b. Determine the inverse of linear and simple non-linear functions, including any necessary restrictions on the domain.

- Determine the inverse of a simple polynomial or simple rational function.
- Identify a logarithmic function as the inverse of an exponential function.

If $x^y = z$, $x > 0$, $x \neq 1$, and $z > 0$, then y is the logarithm to the base x of z . The logarithm $y = \log_x z$ is one of three equivalent forms of expressing the relation $x^y = z$ (the other being $x = \sqrt[y]{z}$).

Example: If $5^a = b$, then $\log_5(b) = a$.

c. Apply properties of logarithms to solve equations and problems and to prove theorems.

- Know and use the definition of logarithm of a number and its relation to exponents.
Examples: $\log_2 32 = \log_2 2^5 = 5$; if $x = \log_{10} 3$, then $10^x = 3$ and vice versa.
- Use properties of logarithms to manipulate logarithmic expressions in order to extract information.
- Use logarithms to express and solve problems.
Example: Explain why the number of digits in the binary representation of a decimal number N is approximately the logarithm to base 2 of N .
- Solve logarithmic equations; use logarithms to solve exponential equations.
Examples: $\log(x - 3) + \log(x - 1) = 0.1$; $5^x = 8$.
- Prove basic properties of logarithms using properties of exponents (or the inverse exponential function).

A.E.5 Relations

a. Know the definition of a relation and distinguish non-function relations from functions.

Example: $x^2 + y^2 = 1$ defines a relation, but not a function.

b. Determine whether a function has the characteristics of reflexivity, symmetry, and transitivity; know that relations exhibiting these characteristics are members of a special class of relations called equivalence relations.

c. Explain how some geometric concepts, such as equality, parallelism, and similarity, can be defined as equivalence relations.

Geometry (G)

Geometry is an ancient mathematical endeavor. The roots of modern geometry were laid by Euclid more than twenty-two centuries ago. Since that time, geometry has been an integral part of mathematics and a common vehicle for teaching the critical skill of deductive reasoning. It offers a physical context in which students can develop and refine intuition, leading to the formulation and testing of hypotheses and ultimately resulting in the justification of arguments, both formally and informally. Geometry also describes changes in objects under such transformations as translation, rotation, reflection, and dilation. It helps students understand the structure of space and the nature of spatial relations. The measurement aspect of geometry provides a basis by which we quantify the world. Geometry is prerequisite for a broad range of activities and leads to methods for resolving practical problems; it can help find the best way to fit an oversized object through a door, aid in carpentry projects, or provide the basis for industrial tool design. Solving practical problems relies to some extent on approximate physical measurements but also rests on geometric properties that are exact in nature. The axioms and definitions of geometry assure us of such measures as the volume of a cube 1 unit on a side, the measure of an angle of an equilateral triangle, or the area of a circle with a given diameter. Grounded in such certainty, geometry provides an excellent medium for the development of students' ability to reason and produce thoughtful, logical arguments.

G.A.1 Angles and triangles

a. Know the definitions and basic properties of angles and triangles in the plane and use them to solve problems.

- Know and apply the definitions and properties of complementary and supplementary angles.
- Know and apply the definitions and properties of interior and exterior angles.

b. Know and prove basic theorems about angles and triangles.

- Know the triangle inequality and verify it through measurement.
In words, the triangle inequality states that any side of a triangle is shorter than the sum of the other two sides; it can also be stated clearly in symbols: If a , b , and c are the lengths of three sides of a triangle, then $a < b + c$, $b < a + c$, and $c < a + b$.
- Verify that the sum of the measures of the interior angles of a triangle is 180° .
- Verify that each exterior angle of a triangle is equal to the sum of the opposite interior angles.
- Show that the sum of the interior angles of an n -sided convex polygon is $(n - 2) \times 180^\circ$.
A common strategy is to decompose an n -sided convex polygon into $n - 2$ triangles.
- Explain why the sum of exterior angles of a convex polygon is 360° .
A possible explanation is that a person walking completely around the perimeter of a convex polygon would have turned through the number of degrees in each external

angle at each vertex and would have made one complete revolution when reaching the starting point again. So, the sum of exterior angles must be 360° .

G.A.2 Rigid motions in the plane

a. Represent and explain the effect of translations, rotations, and reflections of objects in the coordinate plane.

- Identify certain transformations (translations, rotations, and reflections) of objects in the plane as *rigid* motions and describe their characteristics.

Translation, rotation, and reflection move a polygonal figure in the plane from one position to another without changing its linear or angular measurements, i.e. without altering its size or shape.

- Demonstrate the meaning and results of the translation, rotation, and reflection of an object through drawings and experiments.

b. Identify corresponding sides and angles between objects and their images after a rigid transformation.

Corresponding angles are those that lie between edges that correspond under the map showing that the triangles have the same size and shape.

c. Show how any rigid motion of a figure in the plane can be created by some combination of translations, rotations, and reflections.

G.A.3 Measurement

a. Make, record, and interpret measurements.

- Recognize that measurements of physical quantities must include the unit of measurement, that most measurements permit a variety of appropriate units, and that the numerical value of a measurement depends on the choice of unit, and apply these ideas when making such measurements.
- Recognize that real-world measurements are approximations; identify appropriate instruments and units for a given measurement situation, taking into account the precision of the measurement desired.
- Plan and carry out both direct and indirect measurements.

Indirect measurements are those that are calculated based on actual recorded measurements.

b. Apply units of measure in expressions, equations, and problem situations.

- When necessary, convert measurements from one unit to another within the same system.

c. Use measures of weight, money, time, information, and temperature.

- Identify the name and definition of common units for each kind of measurement, e.g., kilobytes of computer memory.

d. Record measurements to reasonable degrees of precision, using fractions and decimals as appropriate.

A measurement context often defines a reasonable level of precision to which the result should be reported.

Example: The U.S. Census bureau reported a national population of 299,894,924 on its Population Clock in mid-October of 2006. Saying that the U.S. population is 3 hundred million (3×10^8) is accurate to the nearest million and exhibits to one-digit precision. Although by the end of that month the population had surpassed 3 hundred million, 3×10^8 remained accurate to one-digit precision.

G.A.4 Length, area, and volume

a. Identify and distinguish among measures of length, area, surface area, and volume.

- Identify and distinguish between the perimeter or circumference of a two-dimensional geometric figure and its area.
- Identify and distinguish between the surface area of a three-dimensional geometric object and its volume.

b. Calculate the perimeter and area of triangles, quadrilaterals, and shapes that can be decomposed into triangles and quadrilaterals that do not overlap.

- Know and apply formulas for the area and perimeter of triangles and rectangles to derive similar formulas for parallelograms, rhombi, trapezoids, and kites.

c. Determine the surface area of right prisms and pyramids whose base(s) and sides are composed of rectangles and triangles.

d. Know and apply formulas for the surface area of right circular cylinders, right circular cones, and spheres.

- Explain why the surface area of a right circular cylinder is a rectangle whose length is the circumference of the base of the cylinder and whose width is the height of the cylinder.

e. Know and apply formulas for the volume of right prisms, right pyramids, right circular cylinders, right circular cones, and spheres.

f. Estimate lengths, areas, surface areas, and volumes of irregular figures and objects.

G.B.1 Angles in the plane

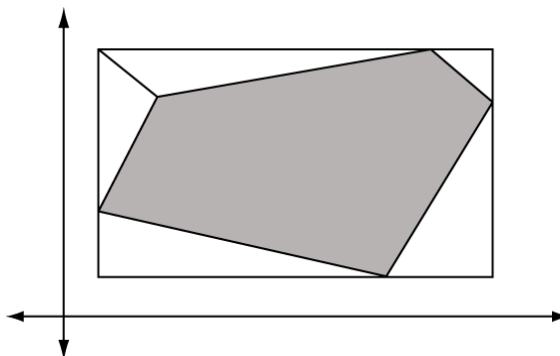
a. Know and distinguish among the definitions and properties of vertical, adjacent, corresponding, and alternate interior angles.

- When a line intersects two parallel lines, identify pairs of angles that are vertical, corresponding, and alternate interior.

- b. Identify pairs of vertical angles and explain why they are congruent.**
- c. Identify pairs of corresponding, alternate interior, and alternate external angles in a diagram where two parallel lines are cut by a transversal, and show that they are congruent.**
- d. Explain why, if two lines are intersected by a third line in such a way as to make the corresponding angles, alternate interior angles, or alternate exterior angles congruent, then the two original lines must be parallel.**
- e. Apply properties of lines and angles to perform basic geometric constructions.**
 - Example: Using only a straight edge and compass, construct the perpendicular bisector of a given line segment and the bisector of a given angle.

G.B.2 Coordinates and slope

- a. Represent and interpret points, lines, and two-dimensional geometric objects in a coordinate plane.**
- b. Determine the area of polygons in the coordinate plane.**
 - Determine the area of quadrilaterals and triangles situated in a coordinate plane whose base/height/length/width are horizontal or vertical line segments.
 - Determine the area of a polygon in the coordinate plane by encasing the figure in a rectangle and subtracting the area of extraneous parts.Example: Find the area of the shaded pentagon by subtracting the area of the five un-shaded triangles from the area of the rectangle shown.



- c. Know how the word slope is used in common non-mathematical contexts, give physical examples of slope, and calculate slope for given examples.**
 - Interpret slope as a physical characteristic.
 - Example: The slope of a ramp with horizontal length 12 feet and vertical rise 4 feet is $4/12$ or $1/3$.
 - Find the slopes of physical objects (roads, roofs, ramps, stairs) and express the answers as a decimal, ratio, or percent.
- d. Calculate the slope of a line in a coordinate plane.**

- Explain how this relates to slope as a physical characteristic.

The measurement of slope in a physical context does not include any inherent concept of positive or negative direction, as is the case with slope in a coordinate plane.

e. Interpret and describe the slope of parallel and perpendicular lines in a coordinate plane.

- Show that the calculated slope of a line in a coordinate plane is the same no matter which two distinct points on the line one uses to calculate the slope.
- Demonstrate why two non-vertical lines in a coordinate plane are parallel if and only if they have the same slope and perpendicular if and only if the product of their slopes is -1 .
- Use coordinate geometry to determine the perpendicular bisector of a line segment.

G.B.3 Pythagorean theorem

a. Interpret and prove the Pythagorean theorem and its converse.

The most common proofs employ geometric dissection.

b. Determine distances between points in the Cartesian coordinate plane.

- Relate the Pythagorean theorem to this process.

c. Apply the Pythagorean theorem and its converse to solve problems.

G.B.4 Circles

a. Identify and explain the relationships among the radius, diameter, circumference, and area of a circle.

- Identify the relationship between the circumference of a circle and its radius or diameter as a direct proportion.
- Identify the relationship between the area of a circle and the square of its radius or the square of its diameter as a direct proportion.
- Explain why the area of a circle equals its radius times one-half of its circumference.

One way to "see" this is to slice a circle into many small pie-shaped pieces and then match them head to toe. The result will be a bumpy rectangular shape whose height is approximately the radius of the circle and whose width is approximately half the circumference of the circle.

b. Show that for any circle, the ratio of the circumference to the diameter is the same as the ratio of the area to the square of the radius and that these ratios are the same for different circles.

- Identify the constant ratio $A/r^2 = \frac{1}{2}Cr/r^2 = C/2r = C/d$ as the number π and show that although the rational numbers 3.14, or $\frac{22}{7}$, or $3\frac{1}{7}$ are often used to approximate π , they are not the actual values of the irrational number π .
- Show that the area of a unit circle (one whose radius is 1) is π .
- Identify and describe methods for approximating π .

Examples: Archimedes constructed a paired sequence of regular polygons with increasing numbers of sides that inscribed and circumscribed a unit circle and used the areas of the polygons to determine upper and lower bounds for the successive approximations for the area of the circle—hence of π ; the Buffon Needle problem provides a method of approximating π using geometric probability—

$$\pi^2 \approx 6 \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right).$$

New computers and software are often tested by calculating π to billions of places.

c. Know and apply formulas for the circumference and area of a circle.

- Determine the perimeter and area of a semicircle and a quarter-circle.

G.B.5 Scaling, dilation, and dimension

a. Analyze and represent the effects of multiplying the linear dimensions of an object in the plane or in space by a constant scale factor, r .

Example: Multiplying the lengths of the sides of a polygon by r results in a polygon having the same shape as the original.

Mathematically, having the same shape means that the image will have the same number of angles and sides as the pre-image, that all angles will preserve their measure, and that corresponding sides will be proportional.

- Use ratios and proportional reasoning to apply a scale factor to a geometric object, a drawing, a three-dimensional space, or a model, and analyze the effect.

b. Describe the effect of a scale factor r on length, area, and volume.

- Explain why triangles in the plane with corresponding sides having a scale factor r have areas related by a factor of r^2 and tetrahedrons in space with corresponding sides having a scale factor r have volumes related by a factor of r^3 .

Since any polygon can be decomposed into a finite number of triangles, it follows that any two similar polygons with scale factor r have areas related by a factor of r^2 . A similar argument works for polyhedral figures in space.

- Extend the concept of scale factor to relate the length, area, and volume of other figures and objects.

Examples: Explain how a change in the length of radius affects the area of a circle and the volume of a sphere; compare the metabolic rate of a man with that of someone twice his size (the metabolic rate of the human body is proportional to the body mass raised to the $\frac{3}{4}$ power).

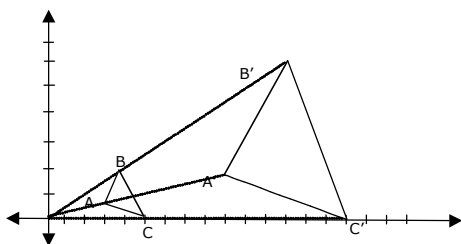
c. Recognize and use relationships among volumes of common solids.

- Recognize and apply the 3:2:1 relationship between the volumes of circular cylinders, hemispheres, and cones of the same height and circular base.
- Recognize that the volume of a pyramid is one-third the volume of a prism of the same base area and height and use this to solve problems involving such measurements.

d. Interpret and represent origin-centered dilations of objects in the coordinate plane.

A dilation centered at the origin with scale factor r maps the point (x, y) to the point (rx, ry) .

Example: In the following figure, triangle $A'B'C'$ with $A'(9,3)$, $B'(12,6)$, and $C'(15,0)$ is the dilation of triangle ABC with $A(3,1)$, $B(4,2)$, and $C(5,0)$. The scale factor for this dilation is 3.



- Analyze the effect on an object when it is subjected to an origin-centered dilation.
The image under an origin-centered dilation will have the same shape as the original object.
- Show that a dilation maps a line to a line with the same slope and that dilations map parallel lines to parallel lines (except for those passing through the origin, which do not change).

G.B.6 Similarity and congruence

a. Interpret the definition and characteristics of similarity for triangles in the plane.

Informally, two geometric objects in the plane are similar if they have the same shape. More formally, having the same shape means that one figure can be mapped onto the other by means of rigid transformations and/or an origin-centered dilation.

- Know that two triangles are similar if their corresponding angles have the same measure.
- Know that the ratio formed by dividing the lengths of corresponding sides of similar triangles is a constant, often called the *constant of proportionality*, and determine this constant for given similar triangles.

b. Apply similarity in practical situations.

- Calculate the measures of corresponding parts of similar figures.
- Use the concepts of similarity to create and interpret scale drawings.

c. Identify and apply conditions that are sufficient to guarantee similarity of triangles.

- Identify two triangles as similar if the ratios of the lengths of corresponding sides are equal (SSS criterion), if the ratios of the lengths of two pairs of corresponding sides and the measures of the corresponding angles between them are equal (SAS criterion), or if two pairs of corresponding angles are congruent (AA criterion).
- Apply the SSS, SAS, and AA criteria to verify whether or not two triangles are similar.
- Apply the SSS, SAS, and AA criteria to construct a triangle similar to a given triangle using straightedge and compass or geometric software.
- Identify the constant of proportionality and determine the measures of corresponding sides and angles for similar triangles.
- Use similar triangles to demonstrate that the rate of change (slope) associated with any two points on a line is a constant.
- Recognize, use and explain why a line drawn inside a triangle parallel to one side forms a smaller triangle similar to the original one.

d. Explain why congruence is a special case of similarity; determine and apply conditions that guarantee congruence of triangles.

Informally, two figures in the plane are congruent if they have the same size and shape. More formally, having the same size and shape means that one figure can be mapped into the other by means of a sequence of rigid transformations.

- Determine whether two plane figures are congruent by showing whether they coincide when superimposed by means of a sequence of rigid motions (translation, reflection, or rotation).
- Identify two triangles as congruent if two pairs of corresponding angles and their included sides are all equal (ASA criterion).

e. Apply the definition and characteristics of congruence to make constructions, solve problems, and verify basic properties of angles and triangles.

- Verify that two triangles are congruent if they are formed by drawing a diagonal of a parallelogram or by bisecting the vertex angle of an isosceles triangle.
- Verify that the bisector of the angle opposite the base of an isosceles triangle is the perpendicular bisector of the base.
- Verify that the base angles of an isosceles triangle are equal.

f. Extend the concepts of similarity and congruence to other polygons in the plane.

A closed plane figure is called a polygon if all of its edges are line segments, every vertex is the endpoint of two sides, and no two sides cross each other.

- Identify two polygons as similar if they have the same number of sides and angles, if corresponding angles have the same measure, and if corresponding sides are proportional.
- Identify two polygons as congruent if they are similar and their constant of proportionality equals 1.
- Determine whether or not two polygons are similar.
- Use examples to show that analogues of the SSS, SAS, and AA criteria for similarity of triangles do not work for polygons with more than three sides.

G.B.7 Visual representations

a. Relate a net, top-view, or side-view to a three-dimensional object that it might represent.

Sufficient information must be given so that "hidden sections" are well-defined.

- Visualize and be able to reproduce solids and surfaces in three-dimensional space when given two-dimensional representations (e.g., nets, multiple views).
- Interpret the relative position and size of objects shown in a perspective drawing.
- Visualize, describe, and identify three-dimensional shapes in different orientations.

b. Draw two-dimensional representations of three-dimensional objects by hand and using software.

- Sketch two-dimensional representations of basic three-dimensional objects such as cubes, spheres, pyramids, and cones.
- Create a net, top-view, or side-view of a three-dimensional object by hand or using software.

c. Visualize, describe, or sketch the cross-section of a solid cut by a plane that is parallel or perpendicular to a side or axis of symmetry of the solid.

G.B.8 Geometric constructions

a. Carry out and explain simple straightedge and compass constructions.

- Copy a line segment, an angle and plane figures; bisect an angle; construct the midpoint and perpendicular bisector of a line segment.

b. Use geometric computer or calculator packages to create and test conjectures about geometric properties or relationships.

G.C.1 Geometry of a circle

- a. Know and apply the definitions and properties of a circle and the radius, diameter, chord, tangent, secant, and circumference of a circle.**
- b. Recognize and apply the fact that a tangent to a circle is perpendicular to the radius at the point of tangency.**
- c. Recognize, verify, and apply statements about the relationships between central angles, inscribed angles and the circumference arcs they define.**

- Show that a triangle inscribed on the diameter of a circle is a right triangle.

- d. Recognize, verify, and apply statements about the relationships between inscribed and circumscribed angles of a circle and the arcs and segments they define.**

An interior angle of a circle is an angle defined by two chords of the circle that intersect in the interior of the circle. An exterior angle of a circle is an angle defined by two chords of the circle that, when extended, intersect outside the circle.

Example: Prove that if a radius of a circle is perpendicular to a chord of the circle, then it bisects the chord.

- e. Determine the length of line segments and arcs, the size of angles, and the area of shapes that they define in complex geometric drawings.**

Examples: Determine the amount of glass in a semi-circular transom; identify the coverage of an overlapping circular pattern of irrigation; determine the length of the line of sight on the earth's surface.

G.C.2 Axioms, theorems, and proofs in geometry

- a. Use geometric examples to illustrate the relationships among undefined terms, axioms/postulates, definitions, theorems, and various methods of reasoning.**

- Analyze the consequences of using alternative definitions for geometric objects.
- Use geometric examples to demonstrate the effect that changing an assumption has on the validity of a conclusion.

- b. Present and analyze direct and indirect geometric proofs using paragraphs or two-column or flow-chart formats.**

- c. Use coordinates and algebraic techniques to interpret, represent, and verify geometric relationships.**

Examples: Given the coordinates of the vertices of a quadrilateral, determine whether it is a parallelogram; given a line segment in the coordinate plane whose endpoints are known, determine its length, midpoint, and slope; find an equation of a circle given its center and radius and conversely, given an equation of a circle, find its center and radius.

- d. Interpret and use locus definitions to generate two- and three-dimensional geometric objects**

Examples: The locus of points in the plane equidistant from two fixed points is the perpendicular bisector of the line segment joining them; the parabola defined as the locus of points equidistant from the point (5, 1) and the line $y = -5$ is

$y = \frac{1}{12}(x - 5)^2 - 2$; the locus of points in space equidistant from a fixed point is a sphere.

- e. Recognize that there are geometries other than Euclidean geometry in which the parallel postulate is not true.**

G.D.1 Triangle trigonometry

- a. Know the definitions of sine, cosine, and tangent as ratios of sides in a right triangle and use trigonometry to calculate the length of sides, measure of angles, and area of a triangle.**
- b. Show how similarity of right triangles allows the trigonometric functions sine, cosine, and tangent to be properly defined as ratios of sides.**
- c. Derive, interpret, and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between 0° and 90° .**

This identity is a special representation of the Pythagorean theorem.

G.D.2 Three-dimensional geometry

- a. Analyze cross-sections of basic three-dimensional objects and identify the resulting shapes.**
- Describe all possible results of the intersection of a plane with a cube, prism, pyramid, or sphere.
- b. Describe the characteristics of the three-dimensional object traced out when a one- or two-dimensional figure is rotated about an axis.**
- c. Analyze all possible relationships among two or three planes in space and identify their intersections.**
- Know that two distinct planes will either be parallel or will intersect in a line.
 - Demonstrate that three distinct planes may be parallel; two of them may be parallel to each other and intersect with the third, resulting in two parallel lines; or none may be parallel, in which case the three planes intersect in a single point, a single line, or by pairs in three parallel lines.

G.E.1 Spherical geometry

- a. Know and apply the definition of a great circle.**

A great circle of a sphere is the circle formed by the intersection of the sphere with the plane defined by any two distinct, non-diametrically opposite points on the sphere and the center of the sphere.

- Show that arcs of great circles subtending angles of 180 degrees or less provide shortest routes between points on the surface of a sphere.

Since the earth is nearly spherical, this method is used to determine distance between distant points on the earth.

b. Use latitude, longitude, and great circles to solve problems relating to position, distance, and displacement on the earth's surface.

Displacement is the change in position of an object and takes into account both the distance and direction it has moved.

- Given the latitudes and longitudes of two points on the surface of the Earth, find the distance between them along a great circle and the bearing from one point to the other.

Bearing is the direction or angle from one point to the other relative to North = 0° . A bearing of $N31^\circ E$ means that the second point is 31° East of a line pointing due North of the first point.

c. Interpret various two-dimensional representations for the surface of a sphere (e.g., two-dimensional maps of the Earth), called projections, and explain their characteristics.

Common projections are Mercator (and other cylindrical projections), Orthographic and Stereographic (and other Azimuthal projections), pseudo-cylindrical and sinusoidal. Each projection has advantages for certain purposes and has its own limitations and drawbacks.

d. Describe geometry on a sphere as an example of a non-Euclidean geometry.

In spherical geometry, great circles are the counterpart of lines in Euclidean geometry. The angles between two great circles are the angles formed by the intersecting planes defined by the great circles.

- Show that on a sphere, parallel lines intersect—that is, the parallel postulate does not hold true in this context.
- Identify and interpret the intersection of lines of latitude with lines of longitude on a globe.
- Recognize that the sum of the degree measures of the interior angles of a triangle on a sphere is greater than 180° .

G.E.2 Vectors

a. Use vectors to represent quantities that have both magnitude and direction.

b. Add and subtract vectors, find their dot product, and multiply a vector by a scalar; interpret the results.

c. Use vectors to describe lines in two- and three-dimensional Euclidean space.

d. Use vectors and their operations to represent situations and solve problems.

e. Use vectors to represent motions of objects in two and three dimensions.

f. Apply parametric methods to represent motion of objects.

G.E.3 Conic sections

a. Develop and represent conic sections from basic properties.

- Know and apply the definitions for an ellipse and a hyperbola.

An ellipse is the locus of all points on the plane the sum of whose distances to two given points, called the foci, is constant. A hyperbola is the locus of all points on the plane the difference of whose distances to two given points, called the foci, is constant.

- Identify a parabola, circle, ellipse, or hyperbola from its equation or key characteristics.

Example: Know that a conic section whose eccentricity is 1 is a circle; identify $x = -4y^2 + 8y + 15$ as a parabola that opens to the left and has an axis of symmetry parallel to the x -axis.

- Derive the equations of circles and parabolas and of ellipses and hyperbolas with axes parallel to the coordinate axes and centered at the origin.
- Describe the effect that changes in the parameters of a particular conic section have on its graph.
- Explain how the key characteristics of standard algebraic forms of ellipses and hyperbolas are related to their graphical characteristics and translate between algebraic and graphical representations.

b. Describe how the intersection of a plane with a cone can form a circle, an ellipse, a parabola, or a hyperbola depending on the orientation of the plane with respect to the axis of the cone.

c. Apply conic sections in modeling real-world phenomena.

Probability and Statistics (PS)

Probability and statistics allow us to make sense of the enormous mass of data that come from measurements of the natural and constructed worlds. Statistical data from opinion polls and market research are integral to informing business decisions and governmental policies. Many jobs require workers who are able to analyze, interpret, and describe data and who can create visual representations of data—charts, graphs, diagrams—to help get a point across succinctly and accurately. Moreover, a free society depends on its citizens to understand information, evaluate claims presented as facts, detect misrepresentations and distortions, and make sound judgments based on available data. Gathering, organizing, representing, summarizing, transforming, analyzing, and interpreting data are essential mathematical skills incorporated in the study of statistics that help fulfill these needs. Statistical reasoning rests on a foundation of probability. The study of probability quantifies the likelihood of an event and includes the analysis of odds and risk and the rigorous prediction of future events.

PS.A.1 Simple probability

a. Represent probabilities using ratios and percents.

b. Compare probabilities of two or more events and recognize when certain events are equally likely.

c. Use sample spaces to determine the (theoretical) probabilities of events.

A sample space consists of all of the disjoint (mutually exclusive) outcomes possible in a given situation involving chance.

- Calculate theoretical probabilities in simple models (e.g., dice, coins, spinners).

d. Know and use the relationship between probability and odds.

The odds of an event occurring is the ratio of the number of favorable outcomes to the number of unfavorable outcomes, whereas the probability is the ratio of favorable outcomes to the total number of possible outcomes.

PS.A.2 Relative frequency and probability

a. Describe the relationship between probability and relative frequency.

If an action is repeated n times and a certain event occurs b times, the ratio $\frac{b}{n}$ is called the relative frequency of the event occurring.

- Recognize and use relative frequency as an estimate for probability.
- Use theoretical probability to determine the most likely result if an experiment is repeated a large number of times.

b. Identify, create, and describe the key characteristics of frequency distributions of both discrete and continuous data.

A frequency distribution shows the number of observations falling into each of several ranges of values; if the percentage of observations is shown, the distribution

is called a relative frequency distribution. Both frequency and relative frequency distributions are portrayed through tables, histograms, or broken-line graphs.

- Describe key characteristics (e.g., shape, symmetry/skewness, typical value, spread) of a frequency distribution.
- Use a probability distribution to assess the likelihood of the occurrence of an event

c. Analyze and interpret actual data to estimate probabilities and predict outcomes.

Example: In a sample of 100 randomly selected students, 37 of them could identify the difference in two brands of soft drinks. Based on these data, what is the best estimate of how many of the 2,352 students in the school could distinguish between the soft drinks?

d. Compare theoretical probabilities with the results of simple experiments (e.g., tossing dice, flipping coins, spinning spinners).

- Explain how the Law of Large Numbers explains the relationship between experimental and theoretical probabilities.

The law of large numbers indicates that if an event of probability p is observed repeatedly during independent repetitions, the ratio of the observed frequency of that event to the total number of repetitions approaches p as the number of repetitions becomes arbitrarily large."

- Use simulations to estimate probabilities.

e. Compute and graph cumulative frequencies.

PS.A.3 Question formulation and data collection

a. Formulate questions about a phenomenon of interest that can be answered with data.

- Recognize the need for data.
- Understand that data are numbers in context (with units) and identify units.
- Define measurements that are relevant to the questions posed.

b. Design a plan to collect appropriate data.

- Understand the differing roles of a census, a sample survey, an experiment, and an observational study.
- Select a design appropriate to the questions posed.
- Begin the use of random sampling in sample surveys and the role of random assignment in experiments, introducing random sampling as a "fair" way to select an unbiased sample.

c. Collect and record data.

- Organize written or computerized data records, making use of computerized spreadsheets.
- Display data on tables, charts, or graphs.
- Evaluate the accuracy of the data.

PS.A.4 Linear trends

- a. Determine whether a scatter plot suggests a linear trend.**
- b. Visually determine a line of good fit to estimate the relationship in bivariate data that suggests a linear trend.**
- Identify criteria that might be used to assess how good the fit is.

PS.B.1 Compound probability

- a. Calculate probabilities of compound events.**
- Employ Venn diagrams to summarize information concerning compound events.
 - Distinguish between dependent and independent events.
- b. Use probability to interpret odds and risks and recognize common misconceptions.**
- Examples: After a fair coin has come up heads four times in a row, explain why the probability of tails is still 50% in the next toss; analyze the risks associated with a particular accident, illness, or course of treatment; assess the odds of winning the lottery or being selected in a random drawing.
- c. Show how a two-way frequency table can be used effectively to calculate and study relationships among probabilities for two events.**
- d. Recognize probability problems that can be represented by geometric diagrams, the number line, or in the coordinate plane; represent such situations geometrically and apply geometric properties of length or area to calculate the probabilities.**

PS.B.2 Analysis and interpretation of categorical and quantitative data

- a. Represent both univariate and bivariate categorical data accurately and effectively.**
- For univariate data, make use of frequency and relative frequency tables and bar graphs.
 - For bivariate data, make use of two-way frequency and relative frequency tables and two-dimensional bar graphs.

b. Represent both univariate and bivariate quantitative (measurement) data accurately and effectively.

- For univariate data, make use of line plots (dot plots), stem-and-leaf plots, and histograms.
- For bivariate data, make use of scatter plots.
- Describe the shape, center, and spread of data distributions. (For bivariate data, a scatter plot may have a linear shape with a trend line marking its center and distances between the data points and the line showing spread.)

c. Summarize and compare data sets by using a variety of statistics.

- For univariate categorical data, use percentages and proportions (relative frequencies).
- For bivariate categorical data, use conditional (row or column) percentages or proportions.
- For univariate quantitative data, use measures of center (mean and median) and measures of spread (percentiles, quartiles, and interquartile range).
- For bivariate quantitative data, use trend lines (linear approximations or best-fit line).
- Graphically represent measures of center and spread (variability) for quantitative data.
- Interpret the slope of a linear trend line in terms of the data being studied.
- Use box plots to compare key features of data distributions.

d. Read, interpret, interpolate, and judiciously extrapolate from graphs and tables.

Extrapolation depends on the questionable assumption that the trend indicated continues beyond the known data.

e. Judge accuracy, reasonableness, and potential for misrepresentation.

- Identify and explain misleading uses of data by considering the completeness and source of the data, the design of the study, and the way the data are analyzed and displayed.

Examples: Determine whether the height or area of a bar graph is being used to represent the data; evaluate whether the scales of a graph are consistent and appropriate or whether they are being adjusted to alter the visual information conveyed.

f. Interpret data and communicate conclusions.

- State conclusions in terms of the question(s) being investigated.
- Use appropriate statistical language when reporting on plausible answers that go beyond the data actually observed.

- Use oral, written, graphic, pictorial, or multi-media methods to create and present manuals and reports.

PS.C.1 Probability distributions

a. Identify and distinguish between discrete and continuous probability distributions.

- Reason from empirical distributions of data to make assumptions about their underlying theoretical distributions.

b. Know and use the chief characteristics of the normal distribution.

The normal (or Gaussian) distribution is actually a family of mathematical functions that are symmetric in shape with scores more concentrated in the middle than in the tails. They are sometimes described as bell shaped. Normal distributions may have differing centers (means) and scale (standard deviation). The standard normal distribution is the normal distribution with a mean of zero and a standard deviation of one. In normal distributions, approximately 68% of the data lie within one standard deviation of the mean and 95% within two.

- Demonstrate that the mean and standard deviation of a normal distribution can vary independently of each other (e.g., that two normal distributions with the same mean can have different standard deviations).
- Identify common examples that fit the normal distribution (height, weight) and examples that do not (salaries, housing prices, size of cities) and explain the distinguishing characteristics of each.

c. Calculate and use the mean and standard deviation to describe the characteristics of a distribution.

d. Understand how to calculate and interpret the expected value of a random variable having a discrete probability distribution.

PS.C.2 Correlation and regression

a. Determine a line of good fit for a scatter plot.

- Identify and evaluate methods of determining the goodness of fit of a linear model (e.g., pass through the most points, minimize the sum of the absolute deviations, minimize the sum of the square of the deviations).
- Use a computer or a graphing calculator to determine a linear regression equation (least-squares line) as a model for data that suggest a linear trend, and understand the criteria it satisfies for goodness of fit.

The linear regression equation for a set of data minimizes the sum of the squared vertical deviations of the points from the line and passes through the point (\bar{x}, \bar{y}) , where \bar{x} is the mean of the x -coordinates of the data points and \bar{y} is the mean of the y -coordinates of the data points. The linear regression line is often called the line of best fit.

- Identify the effect of outliers on the position and slope of the regression line.
- Interpret the slope of the regression line in the context of the relationship being modeled.
- Construct and interpret residual plots to assess the goodness of fit of a regression line.

b. Determine and interpret correlation coefficients.

Be alert to the risk of confusing correlation with causation.

- Recognize correlation as a number between -1 and $+1$ that measures the strength of linear association between two variables.
- Use the relationship among the standard deviations, correlation coefficients, and slope of the regression line to assess the strength of association suggested by the underlying scatter plot.

PS.D.1 Sample surveys, experiments, and observational studies

a. Describe the nature and purpose of sample surveys, experiments, and observational studies, relating each to the types of research questions they are best suited to address.

- Identify specific research questions that can be addressed by different techniques for collecting data.
- Critique various methods of data collection used in real-world problems, such as a clinical trial in medicine, an opinion poll, or a report on the effect of smoking on health.

b. Recognize and explain the rationale for using randomness in research designs.

- Distinguish between random sampling from a population in sample surveys and random assignment of treatments to experimental units in an experiment.

Random sampling is how items are selected from a population so that the sample data can be used to estimate characteristics of the population; random assignment is how treatments are assigned to experimental units so that comparisons among the treatment groups can allow cause-and-effect conclusions to be made.

c. Use simulations to analyze and interpret key concepts of statistical inference.

- Analyze and interpret the notion of *margin of error* and how it relates to the design of a study and to sample size.
- Analyze and interpret the basic notion of *confidence interval* and how it relates to margin of error.
- Analyze and interpret the notion of *p-value* and how it relates to the interpretation of results from a randomized experiment.

d. Plan and conduct sample surveys to estimate population characteristics and experiments to compare treatments.

e. Explain why observational studies generally do not lead to good estimates of population characteristics or cause-and-effect conclusions regarding treatments.

PS.D.2 Risks and decisions

a. Apply probability to practical situations to make informed decisions.

- Communicate an understanding of the inverse relation of risk and return.
- Explain the benefits of diversifying risk.

PS.E.1 Transformations of data

a. Explore transformations of data for the purpose of “linearizing” a scatter plot that exhibits curvature.

- Use squaring, square root, reciprocal, and logarithmic functions to transform data.
- Interpret the results of specific transformations in terms of what they indicate about the trend of the original data.

b. Estimate the rate of exponential growth or decay by fitting a regression model to appropriate data transformed by logarithms.

c. Estimate the exponent in a power model by fitting a regression model to appropriate data transformed by logarithms.

d. Analyze how linear transformations of data affect measures of center and spread, the slope of a regression line and the correlation coefficient.

PS.E.2 Advanced probability

a. Interpret and use the Central Limit Theorem: The distribution of the average of independent samples approaches a normal distribution.

- Use computer simulations to demonstrate the Central Limit Theorem.

b. Know and use equations for the binomial and normal distributions.

c. Use and interpret the normal approximation to the binomial distribution.

d. Calculate and apply expected value.

PS.E.3 Cross-classified data

a. Recognize problems that call for the use of conditional probability and calculate conditional probability in such cases.

- Analyze conditional probabilities using two-way and three-way tables.
- Use and interpret Boolean (and, or, not) operators in the context of two-way and three-way tables.
Example: Student participation on sports teams in relation to Title IX gender equity goals.

b. Use contingency tables to analyze categorical data.

- Understand and illustrate Simpson's Paradox and other data-based paradoxes.

c. Use χ^2 tests to evaluate significance of conditional probabilities.

PS.E.4 Statistical reasoning

a. Explain the protocol for hypothesis testing and apply it in problem situations.

b. Explain statistical estimation and error.

c. Design, conduct, and interpret a simple comparative experiment.

- Formulate questions and identify quantitative measures that may be used to provide answers.
- Draw appropriate conclusions from the collected data.

PS.E.5 Statistical inference

a. Estimate population parameters (point estimators and confidence intervals).

b. Know common tests of significance and use them to test hypotheses.

c. Know and explain the difference between mathematical and statistical inference.